Questions for STEP Section 1 – Division by zero

STEP Mathematics I Summer 2003

2 The first question on an examination paper is:

Solve for x the equation $\dfrac{1}{x} = \dfrac{1}{a} + \dfrac{1}{b}$.

where (in the question) a and b are given non-zero real numbers. One candidate writes $x = a+b$ as the solution. Show that there are no values of a and b for which this will give the correct answer.

The next question on the examination paper is:

Solve for x the equation $\dfrac{1}{x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$.

where (in the question) a , b and c are given non-zero numbers. The candidate uses the same technique, giving the answer as $x = a+b+c$. Show that the candidate's answer will be correct if and only if a , b and c satisfy at least one of the equations $a+b=0$, $b+c=0$ or $c+a=0$.

STEP Mathematics I Summer 2003

6 Evaluate the following integrals, in the different cases that arise according to the value of the positive constant a :

(i) $$\int_0^1 \frac{1}{x^2+(a+2)x+2a}\,dx\ ;$$

(ii) $$\int_1^2 \frac{1}{u^2+au+a-1}\,du\ .$$

STEP Mathematics I Summer 2007

4 Show that $x^3 - 3xbc + b^3 + c^3$ can be written in the form $(x+b+c)\,\mathrm{Q}(x)$, where $\mathrm{Q}(x)$ is a quadratic expression. Show that $2\mathrm{Q}(x)$ can be written as the sum of three expressions, each of which is a perfect square.

It is given that the equations $ay^2+by+c=0$ and $by^2+cy+a=0$ have a common root k. The coefficients a, b and c are real, a and b are both non-zero, and $ac \neq b^2$. Show that

$$\left(ac-b^2\right)k = bc - a^2$$

and determine a similar expression involving k^2. Hence show that

$$\left(ac-b^2\right)\left(ab-c^2\right) = \left(bc-a^2\right)^2$$

and that $a^3 - 3abc + b^3 + c^3 = 0$. Deduce that either $k=1$ or the two equations are identical.

STEP Mathematics I Summer 2003

1 It is given that $\sum\limits_{r=-1}^{n} r^2$ can be written in the form pn^3+qn^2+rn+s , where p , q , r and s are numbers. By setting $n=-1$, 0, 1 and 2, obtain four equations that must be satisfied by p , q , r and s and hence show that

$$\sum_{r=0}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)\ .$$

Given that $\sum\limits_{r=-2}^{n} r^3$ can be written in the form $an^4+bn^3+cn^2+dn+e$, show similarly that

$$\sum_{r=0}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2\ .$$

STEP Mathematics II Summer 2004

1 Find all real values of x that satisfy:

(i) $\sqrt{3x^2+1} + \sqrt{x} - 2x - 1 = 0\,;$

(ii) $\sqrt{3x^2+1} - 2\sqrt{x} + x - 1 = 0\,;$

(iii) $\sqrt{3x^2+1} - 2\sqrt{x} - x + 1 = 0\,.$

STEP Mathematics III Summer 2005

3 Let $\mathrm{f}(x) = x^2 + px + q$ and $\mathrm{g}(x) = x^2 + rx + s$. Find an expression for $\mathrm{f}(\mathrm{g}(x))$ and hence find a necessary and sufficient condition on a, b and c for it to be possible to write the quartic expression $x^4 + ax^3 + bx^2 + cx + d$ in the form $\mathrm{f}(\mathrm{g}(x))$, for some choice of values of p, q, r and s.

Show further that this condition holds if and only if it is possible to write the quartic expression $x^4 + ax^3 + bx^2 + cx + d$ in the form $(x^2 + vx + w)^2 - k$, for some choice of values of v, w and k.

Find the roots of the quartic equation $x^4 - 4x^3 + 10x^2 - 12x + 4 = 0\,.$