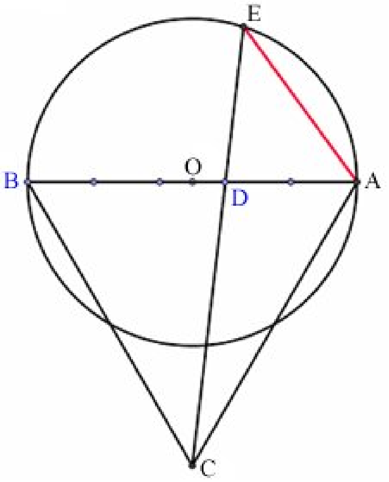
Drawing Polygons

Here is a technical drawing method for drawing a ‘regular’ polygon with *n* sides.

Draw a circle and mark on it a diameter AB. Divide AB into *n* equal parts, using the standard exact construction. Call the 2^{nd} division point from A point D. Draw an equilateral triangle ABC. Join CD and extend it to cut the circle at point P.
Then AE is one side of the polygon.

(i) Show that the method is not exact.
(ii) Explain why it is a good approximation for small values of *n*.
(iii) Investigate the method for large values of *n*.

(i) The method is illustrated for a pentagon below.



Take the radius of the circle to be 1 unit.
Take O to be the origin of a co-ordinate system with OA as the positive *x*-axis.

Then A is the point $(1,0)$ and C is $(0,-\sqrt{3})$.

If the construction is exact, E is the point $(\cos\frac{2\pi}{5},\sin\frac{2\pi}{5})$. Assume this to be the case.

Then CE cuts the *x*-axis at D which is $(1-\frac{4}{5},0)$ or $(0.2,0)$.

The line joining C $(0,-\sqrt{3})$ to E $(\cos\frac{2\pi}{5},\sin\frac{2\pi}{5})$ cuts the *x*-axis at the point $\left(\dfrac{\sqrt{3}\cos(\frac{2\pi}{5})}{\sqrt{3}+\sin(\frac{2\pi}{5})},0\right)$

So the construction is exact if $\dfrac{\sqrt{3}\cos(\frac{2\pi}{5})}{\sqrt{3}+\sin(\frac{2\pi}{5})}=0.2$

In fact $\dfrac{\sqrt{3}\cos(\frac{2\pi}{5})}{\sqrt{3}+\sin(\frac{2\pi}{5})}=0.19948$ to 5 sig. figs.

So the method is not exact.

(ii) Generalising the work in part **(i)** for a *n*-sided polygon rather than a pentagon, the quantities $\dfrac{\sqrt{3}\cos(\frac{2\pi}{n})}{\sqrt{3}+\cos(\frac{2\pi}{n})}$ and $1-\dfrac{4}{n}$ are to be compared.

It can be seen that the method does not apply for $n<3$. It is exact for $n=3$ (the equilateral triangle) and for $n=4$, (the square).

So the cases of interest are those for which $n>4$, although it is actually exact for $n=6$ too.
Let $t=n-4$
Then $1-\dfrac{4}{n}=1-\dfrac{4}{t+4}=\dfrac{t}{t+4}$
This can be expanded as $\dfrac{1}{4}t\left(1+\dfrac{t}{4}\right)^{-1}=\dfrac{t}{4}-\dfrac{t^2}{16}+\dfrac{t^3}{64}-\ldots$ $\quad(t<4$ and so $n<8)$

The equivalent expression $\dfrac{\sqrt{3}\cos\left(\dfrac{2\pi}{n}\right)}{\sqrt{3}+\sin\left(\dfrac{2\pi}{n}\right)}=\dfrac{\sqrt{3}\cos\left(\dfrac{2\pi}{t+4}\right)}{\sqrt{3}+\sin\left(\dfrac{2\pi}{t+4}\right)}=f(t)$ can be expanded as a Maclaurin expansion of $f(t)$: $f(0)=0$.

Differentiating, $f'(t)=\dfrac{2\sqrt{3}\pi\left(\sqrt{3}\sin\left(\dfrac{2\pi}{t+4}\right)+1\right)}{(4+t)^2\left(\sqrt{3}+\sin\left(\dfrac{2\pi}{t+4}\right)\right)^2}$

Substituting $t=0$ gives $f'(0)=\dfrac{\sqrt{3}\pi\left(\sqrt{3}+1\right)}{8\left(\sqrt{3}+1\right)^2}=0.24896$ (to 5 s.f.)

Differentiating again gives $f''(t)$

$$f''(t)=\frac{4\pi\sqrt{3}\left(-(4+t)\left(\sqrt{3}\sin\alpha+1\right)\left(\sqrt{3}+\sin\alpha\right)-\pi\cos\alpha\left(\sqrt{3}+\sin\alpha\right)+2\pi\left(\sqrt{3}\sin\alpha+1\right)\times\sin\alpha\cos\alpha\right)}{(4+t)^2\left(\sqrt{3}+\sin\alpha\right)^3}$$

where $\alpha=\dfrac{2\pi}{t+4}$

And so $f''(0)=\dfrac{-\pi\sqrt{3}}{16\left(\sqrt{3}+1\right)}=-0.12448$ (5 s.f.)

Thus the Maclaurin expansion is $f(t)=f(0)+f'(0)t+\dfrac{1}{2!}f''(0)t^2+\ldots$

$$=0.24896t-0.06224t^2+\ldots$$

This is very similar to $1-\dfrac{4}{t+4}=0.25t-0.0625t^2+\ldots$.

Clearly for small values of *t* the two expressions are close. The similarity of the two expansions, and so the accuracy of the construction, depends on the fact that π can be approximated by $\pi\approx\dfrac{2\left(\sqrt{3}+1\right)}{\sqrt{3}}$. This gives a value of 3.154 for π .

(iii) For large values of *n*, it is appropriate to consider the Maclaurin expansion of

$f(\alpha)=\dfrac{\sqrt{3}\cos\alpha}{\sqrt{3}+\sin\alpha}$ where $\alpha=\dfrac{2\pi}{n}$ $\qquad f(0)=1$.

This gives a series expansion in terms of $\dfrac{1}{n}$ that can be compared to $1-\dfrac{4}{n}$.

$f'(\alpha)=\dfrac{-\sqrt{3}\left(\sqrt{3}\sin\alpha+1\right)}{\left(\sqrt{3}+\sin\alpha\right)^2}$ $\qquad f'(0)=-\dfrac{1}{\sqrt{3}}$

$$f''(\alpha) = \frac{-\sqrt{3}\cos\alpha\left(1-\sqrt{3}\sin\alpha\right)}{\left(\sqrt{3}+\sin\alpha\right)^3} \qquad f''(0) = -\frac{1}{3}$$

$$f(\alpha) = f(0) + f'(0)\alpha + \frac{f''(0)\alpha^2}{2!} + \ldots$$

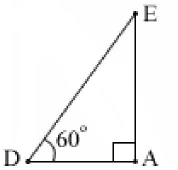
$$\Rightarrow \frac{\sqrt{3}\cos\left(\frac{2\pi}{n}\right)}{\sqrt{3}\sin\left(\frac{2\pi}{n}\right)} = 1 - \frac{1}{\sqrt{3}}\left(\frac{2\pi}{n}\right) - \frac{1}{2}\times\frac{1}{3}\left(\frac{2\pi}{n}\right)^2 + \ldots$$

$$= 1 - 3.6276\times\frac{1}{n} - 6.5797\times\left(\frac{1}{n}\right)^2 + \ldots$$

This is to be compared with $1-\frac{4}{n}$

As $n\to\infty$, the terms in $\left(\frac{1}{n}\right)^2$ and higher powers of $\left(\frac{1}{n}\right)$ may be neglected and so the comparison is between $1-\frac{3.6276}{n}$ and $1-\frac{4}{n}$. Clearly there is substantial difference, indicating 11% error. This means that as $n\to\infty$, the overlap, when a figure with *n* sides is constructed using this method, tends to a limit of about 37^{o}.

Another way to think of the limiting case as $n\to\infty$ is that the angles of the triangle DAE tend to 30^{o}, 60^{o} and 90^{o}.



In this triangle, since $\mathrm{DA}=\frac{4}{n}$ and so $\mathrm{EA}=\frac{4\sqrt{3}}{n}$.

So the perimeter of an *n* sided figure is $n\times\frac{4\sqrt{3}}{n}=4\sqrt{3}$. An arc of length $4\sqrt{3}$ of a circle of radius 1 subtends an angle of 397^{o} (to the nearest degree), so the overlap is 37^{o}.