Double Award GCSE Mathematics

GCSE 2

Exemplar Paper 2
Prereleased material for section B

(Foundation and Higher Tiers)

This is released one month before the examination to allow candidates to work on it with their teachers. A clean copy will be available in the examination.

This paper is one of a set of 6 exemplar papers written by MEI, covering the Foundation and Higher Tiers of GCSE 1 and 2.

The aim of these papers is to inform public discussion. They do not contribute to any existing GCSE qualification.

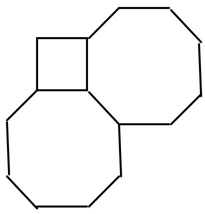
June 2006

Section B: Extended context. Triplets

In this article you will meet two different problems which, at first sight, appear to be unrelated. After spending some time thinking about them you will see that, in fact, they have much in common.

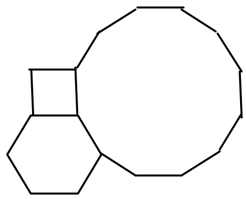
Problem 1: Three regular polygons meeting at a point

Fig.1 and Fig. 2 below each show three regular polygons meeting at a central point.



(4,8,8) sides

Fig.1



(4, 6,12) sides

Fig.2

In order to check if these really do meet exactly, without leaving any gaps or overlapping, you need to know the sizes of the interiors angles in each of these polygons.

Table 3 shows the size of each interior angle in regular 3-, 4-, 5-, 6-, 7- and 8-sided polygons.

Number of sides, n	Size of one interior angle, $180^\circ - \dfrac{360^\circ}{n}$
3	60º
4	90º
5	108º
6	120º
7	$128\frac{4}{7}$º
8	135º
⋮	⋮

Table 3

Now look again at Fig. 1. Around the central point are regular polygons with 4, 8 and 8 sides. Using the values of these interior angles in Table 3 you can see that they add up to 90º+135º+135º=360º. This proves that they really do fit exactly around the central point; there is no gap or overlap.

If you want to find other triplets of regular polygons that fit exactly around a point then you could continue Table 3 for regular polygons with other numbers of sides and look for sets of three interior angles with a sum of 360º. A spreadsheet will allow you to do this quickly.

Alternatively, you could use an algebraic approach.

If three regular polygons with a-sides, b-sides and c-sides fit exactly around a point then

$$\left(180^\circ - \frac{360^\circ}{a}\right) + \left(180^\circ - \frac{360^\circ}{b}\right) + \left(180^\circ - \frac{360^\circ}{c}\right) = 360^\circ .$$

This equation simplifies to

$$180^\circ - \frac{360^\circ}{a} - \frac{360^\circ}{b} - \frac{360^\circ}{c} = 0^\circ$$

which simplifies further to

$$\frac{1}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \qquad \text{[Equation 1]}$$

Substituting values for a, b and c corresponding to the examples in Fig.1 and Fig. 2 above gives

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{2+1+1}{8} = \frac{4}{8} \quad \text{and} \quad \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{3+2+1}{12} = \frac{6}{12}$$

and both of these are equal to $\frac{1}{2}$ as expected.

Now look for other triplets (a, b, c) of regular polygons which meet exactly at a point. You might want to extend Table 3 and find triplets of interior angles that add up to 360º or you might want to find triplets of unit fractions (that is, fractions with a 1 in the numerator) which add up to $\frac{1}{2}$.

How will you be sure that you have found them all?

Problem 2: Equable cuboids

In Fig.1 we used the numbers 4, 8 and 8. In this problem we'll use those numbers again but this time they will be the lengths of the sides of a cuboid. This cuboid is shown in Fig. 4 below.

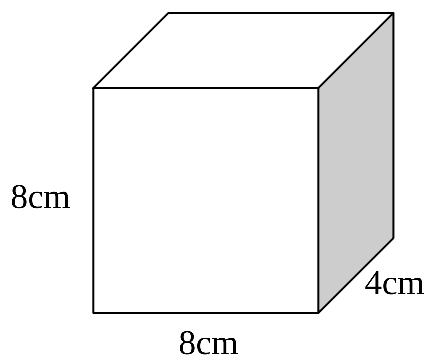


Fig. 4

For this cuboid

- the volume is $4\text{cm} \times 8\text{cm} \times 8\text{cm} = 256\text{cm}^3$
- the surface area is
  $(2 \times 4\text{cm} \times 8\text{cm}) + (2 \times 4\text{cm} \times 8\text{cm}) + (2 \times 8\text{cm} \times 8\text{cm}) = 256\text{cm}^2$

If you ignore the units then the volume of the cuboid is numerically equal to the surface area; they both have a value of 256. Any cuboid with this property, that the volume is numerically equal to the surface area, is called an *equable cuboid*.

You will find that a cuboid with sides of length 4cm, 6cm and 12cm is also equable.

Is it just a coincidence that the two triplets of values which worked in Problem 1 also worked in Problem 2?

To start to answer this question you should check the triplets you found in Problem 1. Do they also work in this problem?

Looking at a few triplets will help but perhaps there are triples you haven't found. A more rigorous approach would be to use algebra.

If the lengths of the sides of an equable cuboid (in cm) are a, b and c then $abc = 2ab + 2bc + 2ca$.

Dividing every term by $2abc$ gives $\frac{1}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

This is exactly the same as Equation 1. Therefore, if you have a triplet that works in Problem 1, it must also work in Problem 2.
In Problem 1, a, b and c had to be positive integers whereas in Problem 2 they could be any positive numbers. This means that although no regular polygon will exactly fill the gap left between a square and a regular nonagon, there is an equable cuboid with two perpendicular edges of length 4cm and 9cm. To find the length of the remaining perpendicular edge you simply need to solve the equation $\frac{1}{2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{x}$.

Although we can show algebraically why these two problems should have the same solution; as far as we know, no-one has been able to give a geometric reason for it yet.