



Further Mathematics Support Programme



www.furthermaths.org.uk

the Further Mathematics Support Programme
www.furthermaths.org.uk




Members only

an introduction to groups

department for education and skills
creating opportunity, releasing potential, achieving excellence

National Centre for Excellence in the Teaching of Mathematics



Mathematics in Education and Industry

Latin squares


A **Latin square** is an $n \times n$ table filled with n different symbols in such a way that each symbol occurs exactly once in each row and exactly once in each column.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3



1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

The great Swiss mathematician Leonhard Euler introduced the idea of latin squares in 1783. He used Latin characters as symbols, hence the name.

Uses:
error correcting codes
Experimental design




Clock arithmetic


$9 + 4 = 1 \pmod{12}$ $9 + 4 = 3 \pmod{10}$

The number $X \pmod{Y}$ is the remainder when X is divided by Y .



Binary operations

Most operations take two elements of a set and combine them to give a definite result; such a rule of combination is called a **binary operation**.




Modular arithmetic

What are the similarities and differences?

+	0	1	2	3	4
0					
1			3		0
2					1
3					
4					3

X	0	1	2	3	4
0					
1			2		
2					3
3					
4					

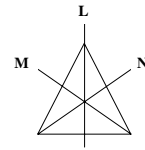
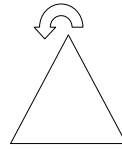


Modular arithmetic – mod 5

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

\times_5	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Symmetry: equilateral triangle



Rotations

- I 0° (or 360°)
- R₁ 120° anticlockwise
- R₂ 240° anticlockwise

Reflections

- L reflection in line L
- M reflection in line M
- N reflection in line N

Groups

		first						
		*	I	R ₁	R ₂	L	M	N
second	I				R ₂			
	R ₁							
	R ₂				R ₁			
	L					I		
	M			L				
	N						R ₁	

Groups

		first						
		*	I	R ₁	R ₂	L	M	N
second	I	I	R ₁	R ₂	L	M	N	
	R ₁	R ₁	R ₂	I	M	N	L	
	R ₂	R ₂	I	R ₁	N	L	M	
	L	L	N	M	I	R ₂	R ₁	
	M	M	L	N	R ₁	I	R ₂	
	N	N	M	L	R ₂	R ₁	I	

Group axioms

A group $(G, *)$ is a non empty set G with a binary operation $*$ such that

- $*$ is **closed** in G
 $a*b \in G$ for all $a, b \in G$
- There is an **identity** element $e \in G$ such that $a*e = e*a = a$ for all $a \in G$
- Inverses**: for all $a \in G$, there is an element a^{-1} such that $a*a^{-1} = e$
- Associativity**:
 $a*(b*c) = (a*b)*c$ for all $a, b, c \in G$

Groups

		first						
		*	I	R ₁	R ₂	L	M	N
second	I	I	R ₁	R ₂	L	M	N	
	R ₁	R ₁	R ₂	I	M	N	L	
	R ₂	R ₂	I	R ₁	N	L	M	
	L	L	N	M	I	R ₂	R ₁	
	M	M	L	N	R ₁	I	R ₂	
	N	N	M	L	R ₂	R ₁	I	

ACCESS TO FURTHER
mathematics

		first			
	*	I	R ₁	R ₂	
second	I	I	R ₁	R ₂	
	R ₁	R ₁	R ₂	I	
	R ₂	R ₂	I	R ₁	

There is one subgroup

MEI

ACCESS TO FURTHER
mathematics

Is it a group?

All integers under subtraction

No – subtraction is not associative

$3 - (6 - 2) = -1$
 $(3 - 6) - 2 = -5$

MEI

ACCESS TO FURTHER
mathematics

Is it a group?

{1, 10} under multiplication modulo 11

X ₁₁	1	10
1	1	10
10	10	1

yes

MEI

ACCESS TO FURTHER
mathematics

Is it a group?

All irrational numbers under multiplication

No – it is not closed

$\sqrt{2} \times \sqrt{2} = 2$, which is rational

MEI

ACCESS TO FURTHER
mathematics

The missing axiom

Commutativity:
 $a * b = b * a$ for all $a, b, \in G$

Is $+_5$ commutative? Yes
 Is \times_5 commutative? Yes
 Is $*$ commutative? No

Groups that are also commutative are called Abelian groups after the Norwegian mathematician Neils Abel

MEI

ACCESS TO FURTHER
mathematics

- The theory of groups started early in nineteenth century in connection with the solutions of algebraic equations.
- Originally a group was the set of all permutations of the roots of an algebraic equation which has the property that combination of any two of these permutations again belongs to the set.
- Later the idea was generalized to the concept of an abstract group. An abstract group is essentially the study of a set with an operation defined on it.
- Group theory has many useful applications both within and outside mathematics.
- Groups arise in a number of apparently unconnected subjects. In fact they appear in crystallography and quantum mechanics in geometry and topology, in analysis and algebra and even in biology.

MEI

A bit of history



Euler, Gauss, LaGrange, Ruffini, Abel Cauchy, Klein, Cayley

Evariste Galois

Born: 25 Oct 1811 in Bourg La Reine (near Paris), France
Died: 31 May 1832 in Paris, France

MEI 2010



PLUS Maths



- <http://plus.maths.org/issue48/package/index.html#intro>
- A package of readings and resources for teachers
- <http://www-history.mcs.st-andrews.ac.uk>
- Find out about your favourite mathematicians

MEI 2010



Here is the ultimate Sudoku variation — Binary Sudoku:

1	

There are 10 possible boards, each more exciting than the last.

0	1
1	0

(of course, that's the binary number 10)

MEI 2010

