Introducing Key Topics in Numerical Methods

MEI Conference 2010

This session will cover

- Motivating student to study NM
- A quick overview of content and good ways to introduce it
- A quick look at coursework
- The tricky bits.

What is Numerical Methods?

How does your calculator work?
It can give you the square root, cube root, n^{th} root, sin, cos, tan, ln, $\sin^{-1}$, $\cos^{-1}$ ….. of any number you put into it.

Does it store all these values in its memory (there are infinitely many)?

There are also many values that you can't ever input into a calculator or computer….

What is Numerical Methods?

One of the earliest approximations is on a Babylonian tablet (approx 1000 – 2000 BC).

It's an approximation to

$\sqrt{2}$

which is accurate to 6 decimal places.

What is Numerical Methods?

Who wants to be a millionaire?
A very famous example is given by the functions governing the motion of a fluid. These are called the Navier-Stokes equations and there is no known solution to them. In fact there is **US$1,000,000** on offer from the Clay Institute to the first person who can find an accurate solution. So get working!

Nonetheless, there are several *approximate* solutions to the Navier-Stokes equations that are known.

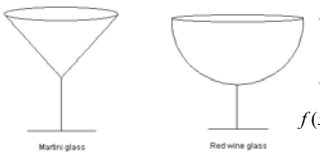
The general form of the Navier-Stokes equations for the conservation of momentum is

$$\rho\frac{D\mathbf{v}}{Dt} = \nabla\cdot\mathbb{P} + \rho\mathbf{f}$$

What is Numerical Methods?

Let's look at a simpler example.

The Martini glass filled to a depth of 6cm contains about $216\pi/3$ of wine. How deep do you need to fill the red wine glass below in order to ensure that it contain the same amount of wine?



The solution is a root of the cubic

$$f(x) = 7\pi x^2 - \frac{x^3}{3} - \frac{216\pi}{3}$$

What is Numerical Methods about?

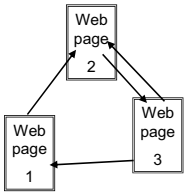
We would rather have an approximate solution to the exact equation rather than an exact solution to the wrong equation.

Numerical Methods are capable of handling non-linearities, complex geometries, very large systems of equations…

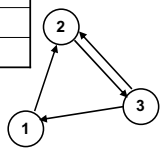
In Engineering

- **Why are Numerical Methods so widely used in Engineering?**
  - Engineers use mathematical modeling (equations and data) to describe and predict behaviour of systems.
  - Closed-form (analytical) solutions are only possible and complete for simple problems (geometry, properties, etc.).
  - Digital computers are widely available, powerful, and cheap.
  - Powerful software packages are available (special or general purpose).

What is Numerical Methods?



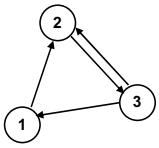
Wepage	Linked to from
1	3
2	1,3
3	2



Suppose the entire world wide web consists of only three webpages. An arrow between two pages in the diagram on the left represents a link between those two pages. For example there is a link from webpage 2 to webpage 3.

We can represent this as a directed graph, see the diagram on the right.

What is Numerical Methods?



Goal is to assign a rank to each webpage.

The rank should be a measure of interest.

Let $\mathbf{x}_1$ be the rank of webpage 1

Let $\mathbf{x}_2$ be the rank of webpage 2

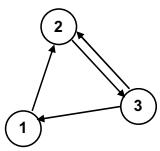
Let $\mathbf{x}_3$ be the rank of webpage 3

We'll also insist that the ranks are normalised, i.e. $x_1 + x_2 + x_3 = 1$

First attempt is to base the rank on the number of incoming links.

Webpage i	Incoming Links	Rank, $\mathbf{x}_i$
1	1	0.25
2	2	0.5
3	1	0.25

What is Numerical Methods?



But surely a good ranking system should take into account the importance of the site that the links are coming from. A link from the www.bbc.co.uk should be worth more than a link from www.barometerworld.co.uk.

In the new model each site has its own rank in voting power. It must distribute this voting power evenly amongst the sites that it links to.

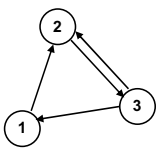
For example site 3 has voting power of $\mathbf{x}_3$. Since it links to sites 1 and 2, it will vote $\mathbf{0.5x}_3$ to site 1 and $\mathbf{0.5x}_3$ to site 3.

Site 1 has voting power of $\mathbf{x}_1$. Since it only links to site 2, it will vote $\mathbf{x}_1$ to site 2.

Site 2 has voting power of $\mathbf{x}_2$. Since it only links to site 3, it will vote $\mathbf{x}_2$ to site 3.

But the total of each site's vote is its rank. This gives the following equations……

What is Numerical Methods?



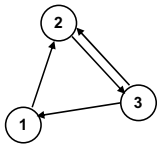
$$x_1 = 0.5x_3$$
$$x_2 = x_1 + 0.5x_3$$
$$x_3 = x_2$$

But this can be written as a matrix equation….

$$\begin{pmatrix} 0 & 0 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The solution of this with $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 = 1$ is $\mathbf{x}_1 = 0.2$, $\mathbf{x}_2 = 0.4$, $\mathbf{x}_3 = 0.4$

What is Numerical Methods?



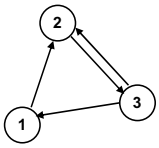
The ranks $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are a solution of

$$\begin{pmatrix} 0 & 0 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The ranks as a column vector are an eigenvector of the matrix with eigenvalue 1 (chosen so that the ranks add up to 1).

How does Google calculate this so quickly? We need to find a quick way to calculate this vector, and hence the ranks.

Iterative Computation



Let $T = \begin{pmatrix} 0 & 0 & 0.5 \\ 1 & 0 & 0.5 \\ 0 & 1 & 0 \end{pmatrix}$

Starting with the vector $x = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$

We look at the sequence of vectors $T\mathbf{x}$, $T^2\mathbf{x}$, $T^3\mathbf{x}$, $T^4\mathbf{x}$,.....

The limit of this sequence is the eigenvector of ranks.

x	Tx	T^2x	T^3x	T^4x	T^5x	T^6x	T^7x	T^8x	T^9x	T^{10}x
0.33	0.167	0.167	0.25	0.17	0.21	0.21	0.19	0.21	0.2	0.2
0.33	0.5	0.333	0.42	0.42	0.38	0.42	0.4	0.4	0.41	0.4
0.33	0.333	0.5	0.33	0.42	0.42	0.38	0.42	0.4	0.4	0.41

What is Numerical Methods?

In the Simpsons episode, Lisa's Sax, two girls at a gifted school play patty-cake while chanting the digits of π :

"Cross my heart and hope to die,
Here's the digits that make π

3.14159265358979323 84...."

This causes real problems for computers

Most of the mathematics you meet is set up so that you can solve it using analytic methods. In the real world many (most) of the equations you meet aren't so easily solvable….

What is Numerical Methods?

Why you can't trust your calculator…

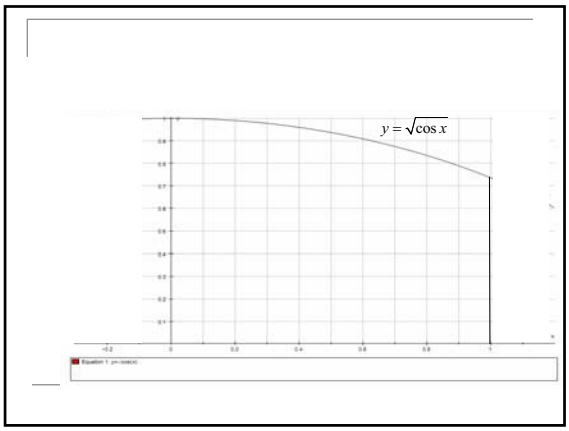
An Equation Flying Past 3D Homer in *Treehouse of Horror VI*

Overview of content

- Numerical Integration
- The Solution of Equation
- Approximating Functions
- Numerical Differentiation
- Error and Rates of Convergence

Numerical Integration

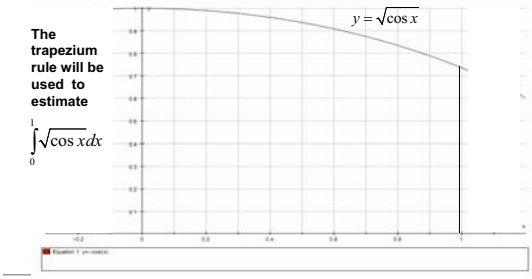
- Nice topic to begin with
- Builds on knowledge students already have.
- Make sense as a natural application of mathematics.



The Trapezium Rule

The trapezium rule will be used to estimate

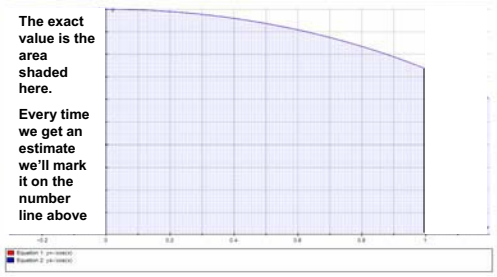
$$\int_0^1 \sqrt{\cos x}\,dx$$



Number Line

The exact value is the area shaded here.

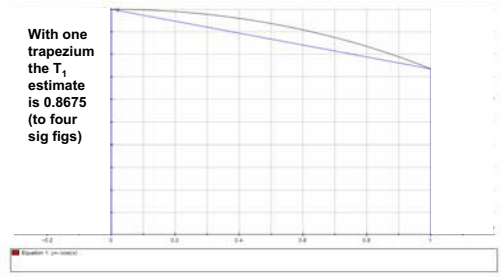
Every time we get an estimate we'll mark it on the number line above



T_1

0.8675

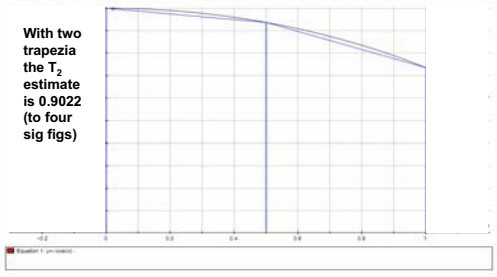
With one trapezium the T_1 estimate is 0.8675 (to four sig figs)



T_1 T_2

0.8675 0.9022

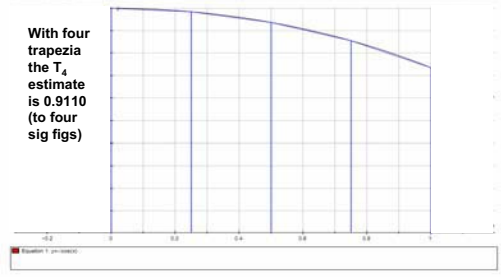
With two trapezia the T_2 estimate is 0.9022 (to four sig figs)

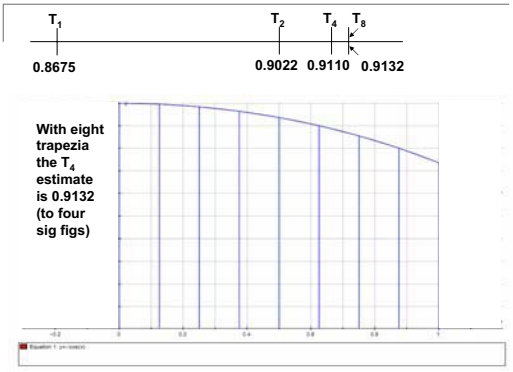


T_1 T_2 T_4

0.8675 0.9022 0.9110

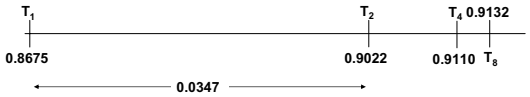
With four trapezia the T_4 estimate is 0.9110 (to four sig figs)





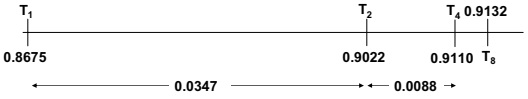
Differences

- Distance between T_2 and T_1 is 0.0347



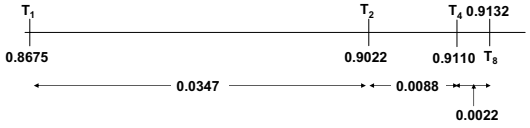
Differences

- Distance between T_4 and T_2 is 0.0088



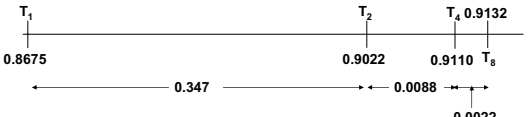
Differences

- Distance between T_8 and T_4 is 0.0022



Ratio of Differences

- How are the gaps reducing?

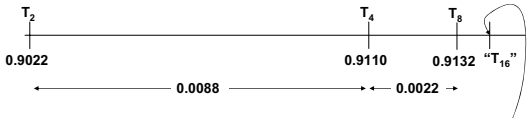


$\frac{0.0088}{0.0347} = 0.2536, \quad \frac{0.0022}{0.0088} = 0.2500$

Each "gap" is about a quarter of the one to its left….

Extrapolation

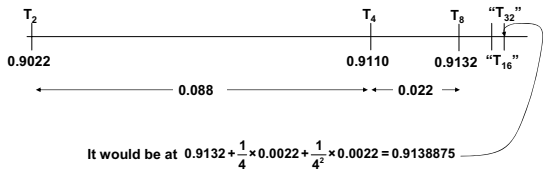
- Where would T_{16} be if this continued?



It would be at $0.9132 + \frac{1}{4} \times 0.0022 = 0.91375$

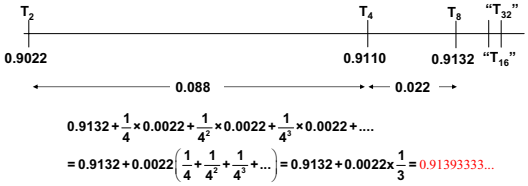
Extrapolation again

- And where would T_{32} be?



Extrapolation – to infinity!

- This sequence tends towards



T1	0.86752629	Gaps	Ratio of	M1	0.936793767	Gaps	Ratio of
T2	0.90216003	0.034633737	Gaps	M2	0.919860815	0.016932952	Gaps
T4	0.91101042	0.008850392	0.255542523	M4	0.915469281	0.004391534	0.259348393
T8	0.91323985	0.002229429	0.251901746	M8	0.914356964	0.001112318	0.253286822
T16	0.91379841	0.000558556	0.250537583	"M16"	0.914078884	0.000278079	0.25
"T32"	0.91393805	0.000139639	0.25	"M32"	0.914009364	0.00006952	0.25
"T64"	0.91397296	0.000034910	0.25	"M64"	0.913991984	0.00001738	0.25
etc	0.91398168	0.000008727	0.25	etc	0.91398764	0.00000434	0.25
etc	0.91398387	0.000002182	0.25	etc	0.913986553	0.00000109	0.25
	0.91398441	0.000000545	0.25		0.913986282	0.00000027	0.25
	0.91398455	0.000000136	0.25		0.913986214	0.00000007	0.25
	0.91398458	0.000000034	0.25		0.913986197	0.00000002	0.25
	0.91398459	0.000000009	0.25		0.913986193	0.00000000	0.25
	0.91398459	0.000000002	0.25		0.913986192	0.00000000	0.25
	0.91398459	0.000000001	0.25		0.913986191	0.00000000	0.25
	0.91398459	0.000000000	0.25		0.913986191	0.00000000	0.25
	0.91398459	0.000000000	0.25		0.913986191	0.00000000	0.25
	0.91398459	0.000000000	0.25		0.913986191	0.00000000	0.25

The Solution of Equation

- Make sure the fundamentals are right first, graphs, table of values, algebra, rearranging equations.
- Use of spreadsheets, what is the right extent to automate?

Approximating Functions

- Approaches to Lagrange's Polynomial.
- Ways to talk about Newton's (forward difference) interpolating polynomial.

Numerical Differentiation

- A great chance to re-do differentiation from first principles.
- Pictures are very useful.
- Extrapolation is possible here too.
- Using rounded values.

Coursework

- Numerical Integration works very well
- Chance to really emphasise some key ideas that are important in the examination.

Numerical Methods (NM) Coursework: Assessment Sheet

Task: Candidates are expected to investigate a problem which is suitable for numerical solution, using one of the methods in the specification.

Coursework Title				
Candidate Name		**Candidate Number**		
Centre Number		**Date**		
Domain	**Mark**	**Description**	**Comment**	**Mark**
Problem specification. (2)	1 1	An appropriate problem is identified. There is a correct explanation of why it is appropriate for numerical solution.		
Strategy. (2)	1 1	An appropriate procedure is used. The candidate explains why the procedure to be adopted is appropriate to the problem.		
Formula application. (3)	1 1 1	There is an application of routine algorithms and methods, which is at least partially correct. There is a substantial application of routine algorithms and methods which is largely correct. The application is fully correct.		
Use of technology. (2)	1 1	Appropriate use is made of calculator or computer software. It is clearly explained.		
Error Analysis. (3)	1	There is a competent analysis of errors, but it may be brief.		

Error Analysis. (3)	1 1 1	There is a competent analysis of errors, but it may be brief. The analysis is developed for the particular problem, either from theoretical formulae or iteration of methods to achieve convergence. This development leads to an improved solution.		
Interpretation. (4)	1 1 1 1	The solution is clearly expressed. The solution is produced to a high degree of accuracy (e.g. 6 significant figures). In addition, there is a discussion of the validity of the solution. Possible limitations are identified.		
Oral communication. (2)	2	Presentation Interview Discussion	Please tick at least one box and give a brief report.	
Half marks may be awarded but the overall total must be an integer. Please report overleaf on any help that the candidate has received beyond the guidelines.			**Total**	18

Authentication should be given on the appropriate sheet, for teachers and candidates
Coursework must be available for moderation by OCR.

NMcoursework/MEIsept2005

Particular points to emphasise in teaching

- Notation in Simpson's Rule
- Numerical Differentiation for small h – loss of significant figures
- Approaches to questions on error
- Detecting order of convergence
- Methods breaking down
- Confusion between secant method and linear interpolation
- Testing knowledge of use of computers