Further Mathematics Support Programme

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How to make Decision Maths exciting

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How to make Decision Maths exciting

know the big ideas

Let Maths take you Further…

What's in D1?

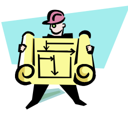
Topic		AQA	Edexcel	MEI	OCR A
Algorithms	Communicating	D1	D1	D1	D1
	Sorting	D1	D1	D1	D1
	Packing		D1	D1	D1
Graphs	Graphs	D1	D1	D1	D1
Networks	Prim	D1	D1	D1	D1
	Kruskal	D1	D1	D1	D1
	Dijkstra	D1	D1	D1	D1
	TSP	D1	D2	D2	D1
	Route inspection	D1	D1	D2	D1
Critical Path Analysis	Activity networks	D2 (on node)	D1 (on arc)	D1 (on arc)	D2 (on arc)
	Cascade charts	D2	D1	D1	D2
Optimisation	Matchings	D1	D1		D2
Linear programming	LP graphical	D1	D1	D1	D1
	LP Simplex	D2	D2	D2	D1
Simulation				D1	

A bit of History

- Decision Maths can often seem like a lot of disconnected ideas put together because they don't fit anywhere else.
- How can you make it into a coherent area of applied maths?
- This session looks at some of the underlying ideas and suggests ways in which the topics can be related to both each other and to other areas of mathematics to make a bigger picture.

What's it about?

- It is probably the most widely used branch of maths in the "real world"
- It is an area of Maths that many students will meet when they go into work



Big Ideas

- Algorithms
- Optimisation
- Operational research
- Mathematical Modelling
- Computers
- Linear Programming
  the glue that holds it all together

Decision making problems

- **Existence**: *does a solution exist?*
- **Construction**: *if a solution does exist, how can you construct a method to find the solution?*
- **Enumeration**: *how many solutions are there? Can you list them all?*
- **Optimisation**: *if there are several solutions, which is the best one? How do you know that this is the best one?*

What is an algorithm?

Construction: *if a solution does exist, how can you construct a method to find the solution?*

Use an algorithm

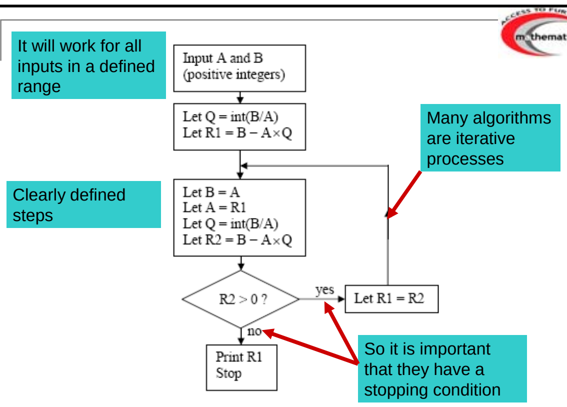
your students already know loads of them

What is an algorithm?

Algorithms must have

- Precision: each step must be well defined
- Generality: it must work for all inputs in a defined range
- Uniqueness: the result at each step will depend only on the inputs and the results of preceding steps
- Finiteness: algorithms must stop after a finite number of steps so they must have a stopping condition.

Many algorithms are iterative processes



90 mins

- Make it relevant – real life examples, use a modelling task (like the old coursework)
- Put it in context – history, links to other areas of mathematics
- Know how it links to other bits of maths
- Make it fun – it lends itself to games, writing your own algorithms etc

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How to make Decision Maths exciting

Put it in context

Know what it's about, where it came from and what it's useful for

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National Centre for Excellence in the Teaching of Mathematics

MEI Mathematics in Education and Industry

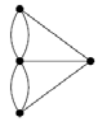
Konigsberg bridges

The **Königsberg bridges** is a famous mathematics problem inspired by an actual place and situation.

The city of Königsberg on the River Pregel in Prussia (now Kaliningrad, Russia) includes two large islands which were connected to each other and the mainland by seven bridges. The citizens of Königsberg allegedly walked about on Sundays trying to find a route that crosses each bridge exactly once, and return to the starting point.

Is it possible to find a route that

- Starts and finishes at the same place?
- Crosses each bridge exactly once?



Euler solves it (again!)

- The paper written by Leonhard Euler on the *Seven Bridges of Königsberg* was published in 1736 is regarded as the first paper in the history of graph theory.
- Euler's formula relating the number of edges, vertices, and faces of a convex polyhedron was studied and generalized by Cauchy and is at the origin of topology.

Update

In 1946 Konigsberg became part of the Soviet Union and it's name was changed to Kaliningrad.

Two of the seven original bridges were destroyed during World War II. Two others were later demolished and replaced by a modern motorway.

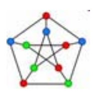
The three other bridges remain, although only two of them are from Euler's time (one was rebuilt in 1935).

Hence there are now only 5 bridges in Konigsberg.

Googlemaps and Google Earth are brilliant tools

Graph theory

- Graph theory was until recently considered a branch of combinatorics, but has grown large enough and distinct enough, with its own kind of problems, to be regarded as a subject in its own right. It has widespread applications in all areas of mathematics and science.



Graphs

Many problems can be modelled as graphs

- circuit diagrams, molecules in chemistry
- The link structure of a website
- The design of silicon chips
- graph theory is also widely used in sociology as a way, for example, to measure an individual's prestige or through the use of social network analysis software.

The development of algorithms to handle graphs is therefore of major interest in computer science and electronics

Networks

- weighted graphs, called networks can be used to represent many different things; for example if the graph represents a road network, the weights could represent the length of each road.
- Network analysis can be used to find the shortest distance between two places or to model and analyse traffic flow

And it's still developing

Robert Prim (pub 1957)

Joseph Kruskal (pub 1956)

Edsgar Dijkstra (D.2002) (pub 1959)

Route inspection (Mei Ko Kwan 1962)

Although some parts are a bit older

- An **Eulerian** Cycle is a closed path that travels along every edge once
- A **Hamiltonian** cycle is a closed path which visits each vertex once and only once.

Games (and abstract algebra)

Sir William Rowan Hamilton, the discoverer of the Hamiltonian Cycle (Route Inspection problem) was Astronomer Royal of Ireland, and a prodigious mathematician.

He invented a puzzle called the Icosian game in 1857.

Hamilton intended that one person should pose the puzzle and a second person solve it.

He sold the rights to toymaker J. Jaques for £25.

The motivation for Hamilton was the problem of symmetries of an icosahedron, for which he invented **icosians**—an algebraic tool to compute the symmetries. The solution of the puzzle is a cycle containing twenty (in ancient Greek *icosa* edges (i.e. a Hamiltonian cycle on the icosahedron).

Links to : symmetry groups

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Make it relevant

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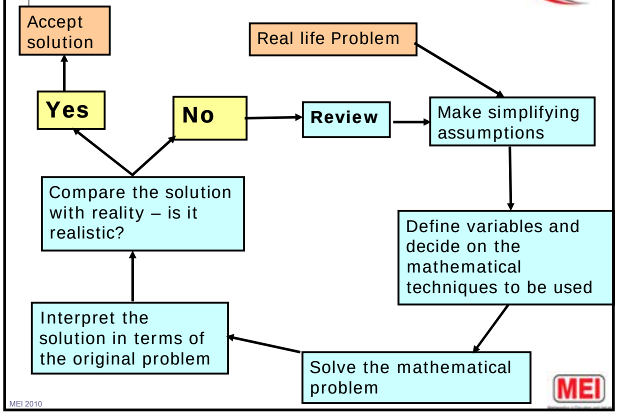
The Modelling Cycle



Modelling exercises

- My favourites
- The travelling weapons inspector
- Opening the deli
- Running a Chinese restaurant

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Make it fun

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DIY algorithms

- Laying cable
- Cooking Breakfast

Play games

- The four colour problem
- Planar graphs

Game Theory
the prisoner's dilemma

Two men are arrested for trying to spend forged. The police inspector in charge of the case believes them both to be counterfeiters so they are taken into different rooms where inspector speaks to each separately

If neither of you confess to counterfeiting we will charge you both with attempting to pass forged notes and you will both get about 2 years in prison.

If you both confess to counterfeiting, we will try to get you a more lenient sentence, probably around 5 years.

If you confess to forgery, but your accomplice does not, we will give you a free pardon but we will charge your friend and he will probably get 8 years.

What should each man do?

the prisoner's dilemma

		Prisoner B			
		confess	refuse	Worst outcome for A (row min)	maximin
Prisoner A	confess	(5, 5)	(0, 8)	5	
	refuse	(8,0)	(2, 2)	8	
Worst outcome for B (column max)		8	2		
minimax					

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Game Theory

- Adam Smith's is reported as saying
  "In competition, individual ambition serves the common good"
- John Nash claims that Smith's theory
  is incomplete, and that "the best result will come from everybody in the group doing what's best for himself, and the group"
- A Beautiful Mind - John Nash

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Minimax/maximin

- Game theory deals with situations where success depends on the choices of others, which makes choosing the best course of action more complex.
- It is an example of a minimax/maximin strategy for solution of problems. Other problems where this is used are
  - Bounds on the TSP problem
  - Dynamic programming

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Investigating Bin Packing

Divide a group of 5 weights (2, 2, 2, 3 and 3) into two piles so that each pile is as close as possible in total weight.

By inspection

3, 3

2, 2, 2

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Investigating Combinatorial Mathematics

Ronald Graham (Bell laboratories) developed this algorithm for packing weights most efficiently:

Starting with the heaviest weight and working down to the lightest, put each weight into the pile that tends, at each step of the way, to keep the weights of the piles as equal as possible.

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Investigating Combinatorial Mathematics

2, 2, 2, 3 and 3

Using Graham's algorithm to solve the problem we get:

3, 2, 2
3, 2

This is **not the best solution**, but it is also **not the worst combination**, which it would have been if piles ranged in size from 2 to 10

3, 3, 2, 2, 2

OR 3, 3, 2, 2 and 2

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The best solution vs the algorithmic solution

The best solution is 3, 3 and 2, 2, 2.

Graham's algorithm gives a solution that is out by 1/6 or about 16%.

Graham was able to prove that for 2 piles and any distribution of any number of weights, his algorithm will never be off by more than 16%.

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Bin packing algorithms

Full bin – not practical for large numbers of objects

First fit - fit things into the first available bin that will take them

First fit decreasing – put the items in order of size then fit them into the first available bin that will take them

FFD ought to be better

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Your problem

1. You have 33 weights and bins with a capacity of 524 weight units. Using the first-fit decreasing algorithm, divide up the blocks provided into as few bins as possible.
2. Now remove the 46 and repeat the algorithm. What happens?

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Solution 1.

Bin 1	442	46	12	12	12	
Bin 2	252	252	10	10		
Bin 3	252	252	10	10		
Bin 4	252	252	10	10		
Bin 5	252	127	127	9	9	
Bin 6	127	127	127	106	37	
Bin 7	106	106	106	85	84	37

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Solution 2.

Bin 1	442	37	37					
Bin 2	252	252	12					
Bin 3	252	252	12					
Bin 4	252	252	12					
Bin 5	252	127	127	10				
Bin 6	127	127	127	106	10	10	10	
Bin 7	106	106	106	85	84	10	10	9
Bin 8	9							

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Food for thought

- In 1973, Jeffrey Ullman of Princeton University showed that the first-fit packing algorithm can be off by as much as 70%!
- First-fit decreasing is never more that 22% off.
- In 1973 David Johnson (a colleague of Graham's at Bell labs) proved that, in general FFD cannot be beaten (the proof takes 75 pages)

But is still throws up anomalies ….

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Know about the Big ideas
Optimisation

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Nov 2009 - Feb 2010

You can't win them all

- The **Travelling Salesman Problem** (**TSP**) was first formulated as a mathematical problem in 1930 and is one of the most intensively studied problems in optimisation.
- The TSP has several applications even in its purest formulation, such as planning, logistics, and the manufacture of microchips.
- Even though the problem is computationally difficult, a large number of heuristics and exact methods are known, so that some instances with tens of thousands of cities can be solved.
- BUT we have no clever algorithm for solving it

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How good is your algorithm?

The efficiency of an algorithm is measured by it's **complexity.**

- In the theory of computational complexity, the TSP problem belongs to the class of **NP-complete** (nondeterministic polynomial time) problems.
- Although any given solution to such a problem can be verified quickly, there is no known efficient way to locate a solution in the first place.
- The time required to solve the problem increases very quickly as the size of the problem grows. As a consequence, determining whether or not it is possible to solve these problems quickly is one of the principal unsolved problems in computer science today.
- It is likely that the worst case running time for TSP increases exponentially with the number of cities.

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Linear Programming
the glue that holds it all together

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Finding the Optimum Value

Graphing inequalities is in GCSE

constraints

$2x + y \leq 16$ Lathe

$2x + 3y \leq 24$ Assembler

Profit Line

Draw a line through the origin parallel to the gradient of the profit function. Move this line up the y-axis until it is just leaving the feasible region – the point at which it leaves the feasible region is the optimum value.

Profit line
$y = 1/14\ (P - 16x)$

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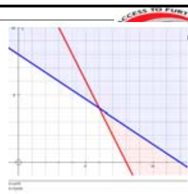
Finding the Optimum Value

Method 1: Tour of vertices

(0,8) profit = £112
(6,4) profit = £152
(8,0) profit = £128
Optimal solution is to make 6 bicycles and 4 trucks. Profit £152



constraints

$2x + y \leq 16$ Lathe

$2x + 3y \leq 24$ Assembler

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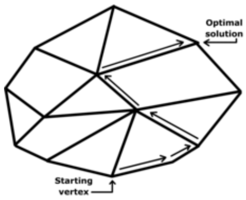
What happens if there are more than two variables?

In geometric terms we are considering a closed, convex, region, *P*, (known as a polytope), defined by intersecting a number of half-spaces in *n*-dimensional Euclidean space (these are the constraints).

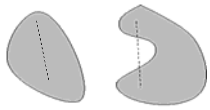
If the objective is to maximise a linear function *L*(*x*), consider the family of hyperplanes, *H*(*c*), defined by *L*(*x*) = *c*.

As *c* increases, these form a parallel family. We want to find the largest value of *c* such that *H*(*c*) intersects *P*.



In this case we can show that the optimum value of *c* is attained on the boundary of *P* using the *extreme point theorem*

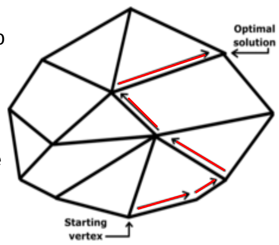
If *P is* a convex polygon and *L*(*x*) is a linear function then all of the values of *L*(*x*) at the points of *P*, both maximum and minimum occur at the extreme points.
Hence, if an LP has a bounded optimal solution then there exists an extreme point of the feasible region that is optimal



An Introduction to Linear Programming and the Theory of Games by Abraham M Glicksman
Published by Dover publications Isbn 0-486-41710-7

Introducing the simplex method

Methods for finding this optimum point on P work in several ways: some attempt to improve a possible point by moving through the interior of *P* (so-called *interior point methods*);
others start and remain on the boundary searching for an optimum.



Introducing the simplex method

The simplex algorithm follows this latter methodology.

Start at some vertex of the region, and at every iteration we choose an adjacent vertex such that the value of the objective function does not decrease. If no such vertex exists, we have found a solution to the problem.

But usually, such an adjacent vertex is non-unique, and a *pivot rule* must be specified to determine which vertex to pick (various pivot rules exist).

What is linear programming?

In this case we can show that the optimum value of *c* is attained on the boundary of *P* using the *extreme point theorem*

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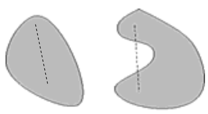

What is linear programming?

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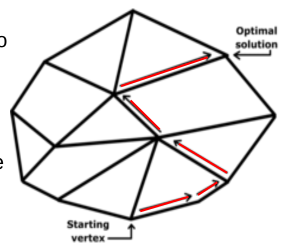

Linear Programming

- Linear programming is probably the single most used mathematical method in the world at the current time.
- Almost all the examples here and many more can be converted into LP problems that can be solved by computer

The simplex algorithm uses matrix techniques (Gauss_Jordan Elimination) to solve series of simultaneous equations in many unknowns

Some examples for students and teachers

- **Business:** Scheduling using Critical Path analysis
- **Nutrition**: optimal mix of ingredients to ensure adequate nutrition for minimum cost
- **Logistics**: transporting goods efficiently (shortest distance, minimum costs etc)
- **Finance:** Lowest bid - electronic auction
- **Health**: Nurse scheduling, reducing queuing times

These examples and others can be found on the OR Society website: http://www.learnaboutor.co.uk/

O.R. Inside F1.