MEI Conference, July 2007

Notes on *Six gems in mechanics*

M1 Suppose that a number of fireworks are fired from ground level and travel over horizontal ground. Suppose also that air resistance is negligible.

No matter what their initial speeds and angles of projection, all the fireworks that reach their highest points at the same time do so at the same height. [If the fireworks turned into flares when they reached their highest points you would see a rising horizontal plane of light.]

If the greatest height reached is H m reached in T s, then taking + ve as upwards and using $s = vt - \frac{1}{2}at^2$ gives $H = 0 - \frac{1}{2}(-g)T^2$ so $H = \frac{1}{2}gT^2$, which does not require separate knowledge of the speed, u, and angle of projection, α. This is because both H and T depend only on the vertical component of the initial velocity, $\dot{y}_0$, and $\dot{y}_0 = u\sin\alpha$: i.e. it is the value of $u\sin\alpha$ that determines both H and T. [Considering the vertical direction, we have, using $v = u + at$, that $0 = \dot{y}_0 - gT$ so $T = \dfrac{\dot{y}_0}{g}$ and, using $v^2 = u^2 + 2as$, that $0 = \dot{y}_0^2 - 2gH$ so $H = \dfrac{\dot{y}_0^2}{2g}$.]

M1 Two misapprehensions are commonly held about the properties of the two trajectories that pass through a point inside a parabola of safety. These are:

> one is rising and one is falling,
>
> the angle of projection of one is greater than 45° and that of the other is less than 45°.

That these statements are false is clearly shown in an Autograph demonstration.

Suppose that particles are projected with the same speed from the same point in the same vertical plane but have different angles of projection. It is well known that the plane containing the trajectories may be partitioned into three regions: points on a parabola (called the parabola of safety) where there is a single trajectory that passes through that point: points inside that parabola for each of which there are two distinct trajectories that pass through them: points outside the parabola for each of which there are no trajectories that pass through them. This is easy to derive and even easier to show using Autograph.

The parabola of safety may be found from the equation of the trajectory as follows. Using the usual symbols, we make $\tan\alpha$ the subject of the trajectory equation to give $\frac{gx^2}{2u^2}\tan^2\alpha - x\tan\alpha + y + \frac{gx^2}{2u^2} = 0$. We require equal roots for a single solution so $x^2 = 4\times\frac{gx^2}{2u^2}\left(y+\frac{gx^2}{2u^2}\right)$ giving $y = -\frac{g}{2u^2}x^2 + \frac{u^2}{2g}$. This parabola intersects the $y-$axis at $\left(0,\frac{u^2}{2g}\right)$ and the +ve $x-$axis at $\left(\frac{u^2}{g},0\right)$.

No two of the trajectories can intersect if they are both rising so either one is rising and one is falling or they are both falling. If one is rising and the other falling, the point of intersection must be before the highest point of one of the trajectories. The locus of the highest points may be found as follows:

with the usual conventions of the origin at the point of projection and initial speed and angle of u and α, respectively, the coordinates of the highest point are easily shown to be $\left(\frac{u^2\sin 2\alpha}{2g},\frac{u^2\sin^2\alpha}{2g}\right)$.

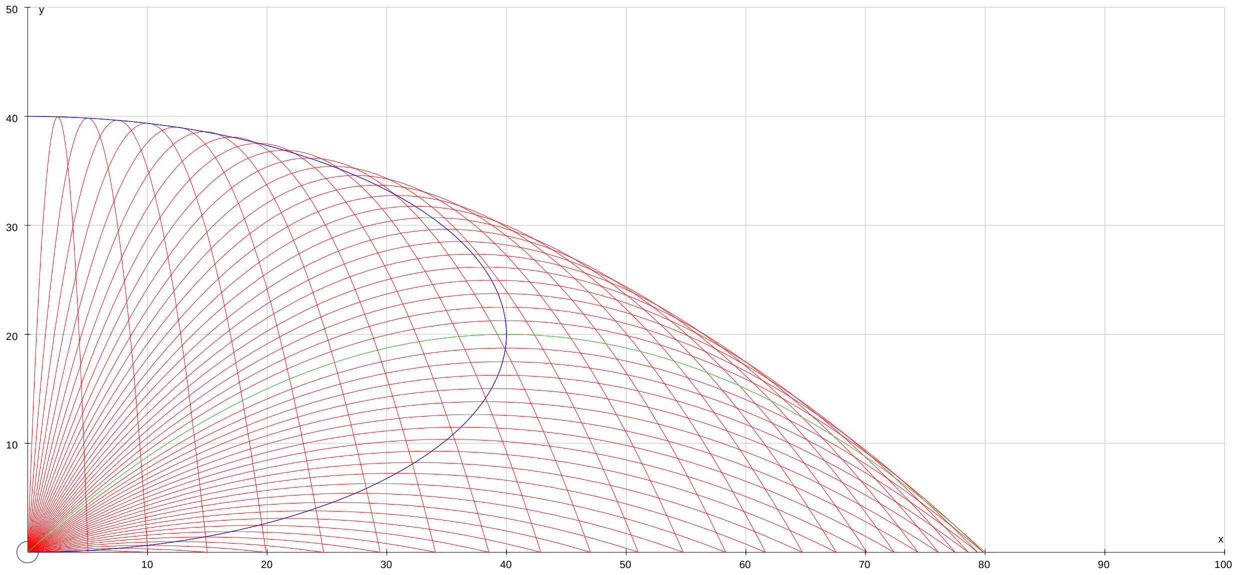
Setting $x = \frac{u^2\sin 2\alpha}{2g}$ and $y = \frac{u^2\sin^2\alpha}{2g} = \frac{u^2}{4g}(1-\cos 2\alpha)$ we have $\sin 2\alpha = \frac{2gx}{u^2}$ and $\cos 2\alpha = 1 - \frac{4gy}{u^2}$.

Eliminating α gives $\frac{4g^2x^2}{u^4} + \left(1-\frac{4gy}{u^2}\right)^2 = 1$ so $\frac{4g^2x^2}{u^4} + \frac{16g^2}{u^4}\left(\frac{u^2}{4g}-y\right)^2 = 1$.

This is an ellipse with semi – major axis $\frac{u^2}{2g}$ and semi – minor axis $\frac{u^2}{4g}$ and with centre $\left(0, \frac{u^2}{4g}\right)$.

Also, no two trajectories can intersect if they both have angles of projection less than 45° so either one is less than 45° and one greater or they are both greater and the trajectory with angle of projection of 45° is the boundary between these cases.

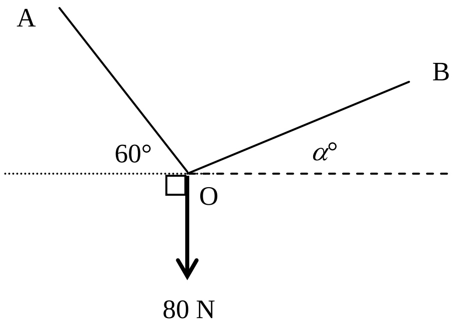
The regions of the plane correspond to the four possible combinations can be seen as those inside the parabola of safety partitioned by the ellipse found above and the trajectory with angle of projection of 45°.



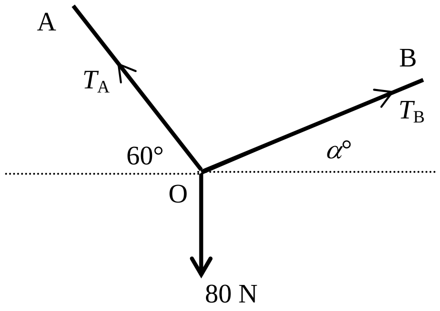
[This diagram was obtained using Autograph taking ‘*u*’ as 28.
Degrees mode selected.
The parameter *t* is set as ‘manual’ from 0 to 100 step 0.1.
The family is $x = 28t\cos\alpha,\ y = 28t\sin\alpha - 4.9t^2$ with ‘Family’ selected for α start 0 finish 90 and ‘number’ 40.
The ellipse is $x = 40\sin 2t,\ y = 40\sin^2 t$
The trajectory initially at 45° is $x = 28t\cos 45,\ y = 28t\sin 45 - 4.9t^2$.]

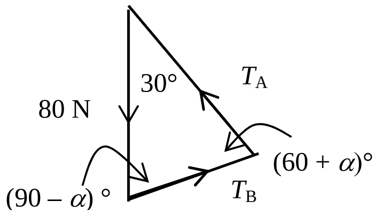
M1 Representation of static equilibrium with a force triangle can give ready insights into how the forces depend on the geometry of the system. In particular, consider a particle held in equilibrium by two light strings with one string at a constant angle and the other at varying angles.

The diagram shows a particle of weight 80 N suspended at O by two light strings OA and OB with AO at 60° to the horizontal. We are asked to find the tension in each of the strings for different values of α.



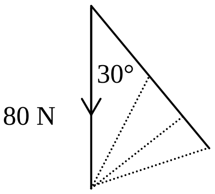
The force diagram and force triangle are shown below.





For any α, the sine rule (or simple trigonometry if $\alpha = 30$) applied to the force triangle provides a quick and efficient solution to the problem.

Further relationships between the angles of the strings and the tensions in them can readily be seen from the force triangle. If the angle α is increased from 0, then the magnitude of T_B decreases until the strings are perpendicular and then it begins to increase again. Meanwhile, the magnitude of T_A decreases.



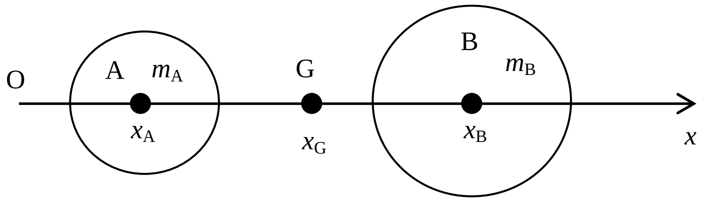
These relationships are much harder to see using the equations for horizontal and vertical equilibrium which are $T_A \cos 60 = T_B \cos\alpha$ and $T_A \sin 60 + T_B \sin\alpha = 80$ respectively. These give $T_B = \dfrac{80}{\sqrt{3}\cos\alpha + \sin\alpha}$ which cannot be minimised analytically with AS level skills. Even the ‘nice’ angle of 60° did not help.

The force triangle method applied to the most general case is always within the AS syllabus.

M2 Both an empty and a full can of lager have their centre of mass (approximately) in the middle – the approximation being because the can may not be completely full and there is a ring-pull. When you drink a little lager, the centre of mass clearly moves towards the base. When is it at its lowest point?

This may be generalised to ask what level of liquid is required in any container to give the lowest possible combined centre of mass of the container and liquid. We shall use an approach that characterises the solution for any shape of container and density of the liquid.

We start by observing that the overall centre of mass of fixed objects A and B lies on the line segment joining the centres of mass of A and B. Suppose the objects lie on an x – axis with masses and positions as shown in the diagram.



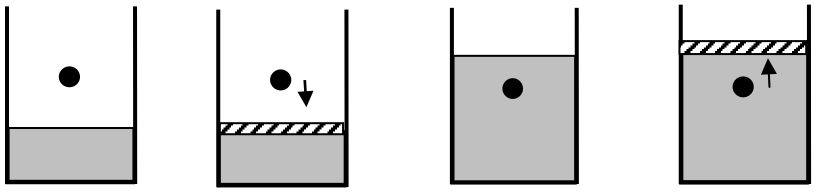
We have $\left(m_A + m_B\right) x_G = m_A x_A + m_B x_B \Rightarrow x_G = \dfrac{m_A x_A + m_B x_B}{m_A + m_B}$.

Since $x_B > x_A$ we have $x_G > \dfrac{m_A x_A + m_B x_A}{m_A + m_B} = x_A$.

If we think of object B as being added to an existing system A, then we observe that the effect is that the position of the combined c.m. has been moved from the position of the c.m. of A towards the position of the c.m of B.

Now consider adding further liquid to the liquid present (if any) in the two cases
- (a) the level of the liquid is below the combined centre of mass of the container and liquid,
- (b) the level of the liquid is above the combined centre of mass of the container and liquid.

These cases are shown in the diagram.



(a) (b)

● position of combined centre of mass

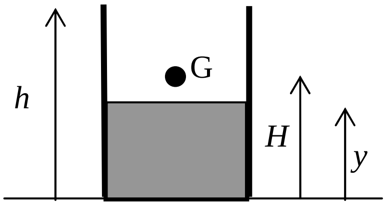
▨ added liquid

In case (a) the combined c.m. moves downwards and in case (b) upwards. We can see that if we start with an empty container and slowly add liquid, the c.m. descends while the combined c.m. is above the top surface of the liquid and rises when it is below it.

Clearly the lowest position of the combined c.m. is when it is in the top surface of the liquid. Equally clearly, this characterisation of the position depends neither on the shape of the container nor on the relative masses of the container and liquid etc.

Example

An open cylindrical can has height h cm, its c.m. at G, H cm above the base, and a mass of m kg. The can when full holds M kg of liquid.



When the height of the liquid is y cm, the c.m. of the can and liquid, $\bar{y}$, is given by

$$\frac{y}{h}M \times \frac{y}{2} + mH = \left(\frac{y}{h}M + m\right)\bar{y} \qquad (1).$$

Suppose the lowest position of the combined c.m. of the can and liquid is Y cm above the base.
From the result above, when $y = Y$ we have $\bar{y} = Y$ giving $MY^2 + 2mhY - 2mhH = 0$.

[The alternative to this approach is to use calculus on (1) to minimise $\bar{y}$ as a function of y, a much more demanding enterprise.]

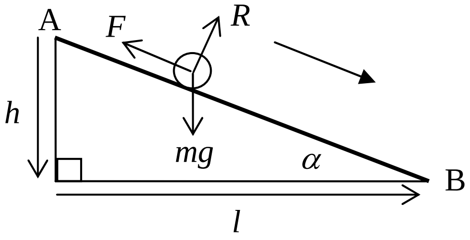
Suppose we take the case when $M = 4m$ and $H = \frac{3}{8}h$.

This gives $4mY^2 + 2hmY - 2hm \times \frac{3}{8}h = 0$ so $16Y^2 + 8hY - 3h^2 = 0$

This gives $(4Y - h)(4Y + 3h) = 0$ so, taking the positive value, $Y = \frac{h}{4}$.

M2 Suppose that a particle of mass m moving in a vertical plane slides down a uniformly rough slope. Suppose that the vertical displacement downwards is H, the horizontal distance travelled is L and the coefficient of friction is μ. If the path of the particle is any smooth curve and the particle is at all times in contact with the slope, the KE gained by the particle is $mg(H-\mu L)$.

Many problems easily solved by energy methods are tackled by students who falsely (or without justification) modify the problem so that may apply constant acceleration formulae. This is a case where it works!



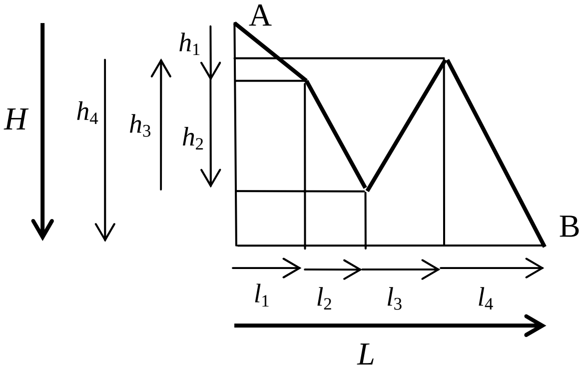
To start with, suppose a particle of mass m slides down a small straight section of plane AB, as shown in the diagram. We shall neglect air resistance and assume that the coefficient of friction between the particle and the plane has a constant value of μ and the particle never loses contact with the plane. The vertical and horizontal displacements of the particle in travelling from A to B are h and l respectively. The normal reaction is R and the frictional force F.

Since the particle is sliding, we have $F = F_{\max} = \mu R = \mu mg\cos\alpha$.
The KE gained by the particle as it travels from A to B, ΔKE, is given by $\Delta\text{KE} = \text{AB}\times mg\sin\alpha - \text{AB}\times\mu mg\cos\alpha = mgh - \mu mgl$ since $\text{AB}\sin\alpha = h$ and $\text{AB}\cos\alpha = l$.

This gives $\Delta\text{KE} = mg\left(h-\mu l\right)$.

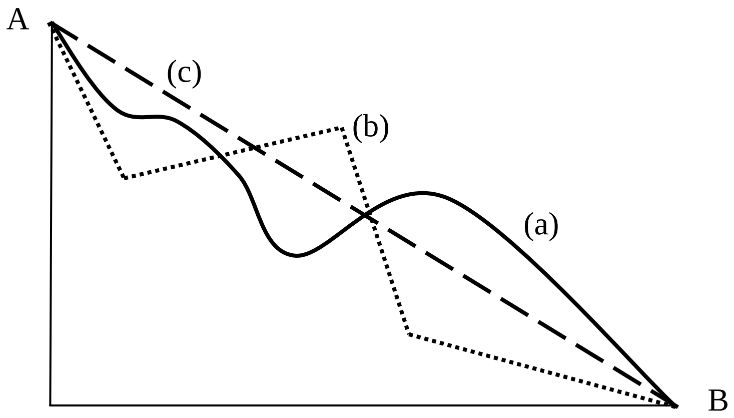
Note that this expression is also valid for the particle sliding *up* the slope if α is taken to be negative; the displacement h is now negative while the displacement l remains positive. This means that we can apply this result to any sequence of such sections as shown below. If the vertical displacements are h_1, h_2, h_3, … and the horizontal displacements l_1, l_2, l_3, … then if $H=\sum h_i$ and $L=\sum l_i$ we have $\Delta\text{KE} = mg\left(H-\mu L\right)$.

Of course, we are assuming that the particle has enough KE initially for it to travel from A to B (i.e. get over any bumps) and that it never loses contact with the slopes.



We may make the straight line segments as small as we like and so move to a smooth curve.

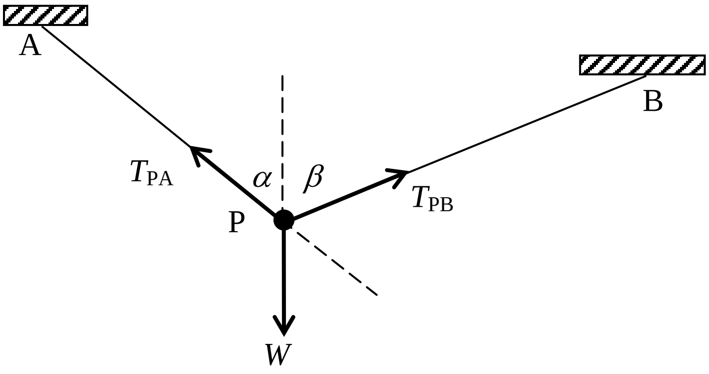
It is, perhaps, not obvious that a particle would gain exactly the same speed sliding from A to B down any of the curves (a), (b) and (c). All that is necessary is that the slope is uniformly rough – the length and shape of the path from A to B do not matter.



M3 A small object is held in static equilibrium by the use of two light strings. What happens to the tension in one string when the other is cut?

Two light strings AP and BP are fixed at A and at B and are attached to a particle of weight W at P. The system is in equilibrium.

We first find the tension in AP.



Resolving horizontally gives $T_{\text{PA}} \sin\alpha = T_{\text{PB}} \sin\beta$

Resolving vertically gives $T_{\text{PA}} \cos\alpha + T_{\text{PB}} \cos\beta = W$

Solving for $\beta \neq 0$ gives $T_{\text{PA}} = \dfrac{W \sin\beta}{\sin(\alpha+\beta)}$ (1)

Suppose that the string PB is cut. The system is no longer in equilibrium but the particle now swings in a circular arc centre A starting from rest. (i.e. like a pendulum at the top of its swing).

The equation of motion in the radial direction is $W\cos\alpha - T_{\text{PA}} = -\dfrac{mv^2}{\text{AP}}$, where m is the mass of the particle and v is its speed.

At the instant the string is cut, $v = 0$ so $T_{\text{PA}} = W\cos\alpha$ (2)

We can see that typically (1) and (2) have different values so there is an instantaneous change in the tension in the string AP.

When is there no change in tension?

This can be seen readily as follows. We know what the tension in AP must be after the string is cut so we need the same tension before that happens. All we need is no component of the tension in PB in the direction of AP, i.e. $\alpha + \beta = 90°$. Checking in (2) we get $T_{\text{PA}} = \dfrac{W\sin(90-\alpha)}{\sin 90} = W\cos\alpha$ as required.

[For reference, resolution in the direction AP before the string is cut gives $W\cos\alpha + T_{\text{PB}}\cos(\alpha+\beta) - T_{\text{PA}} = 0$.

It is worth noting that result (1) comes immediately from a force triangle and so is in the AS syllabus. The approach above requires much higher level manipulative skills with trigonometric identities.]