

# Four Colour Theorem

## 1. History

1852: Can every map be coloured with at most four colours in such a way that neighbouring countries are coloured differently?

Early 'proofs' attempted to show that five countries could not each border other four. This is impossible but does not constitute a proof.

## 2. Euler's theorem

$F - E + V = 2$  Prove by reducing maps by removing edges.

## 3. Cubic maps

We can restrict our search to cubic maps, i.e. maps which always have three boundary lines meeting at any vertex.

## 4 Counting formula

If number of countries with  $n$  boundary edges is  $C_n$  then

$$F = C_2 + C_3 + C_4 + \dots + C_n$$

$$2E = 2C_2 + 3C_3 + 4C_4 + \dots + nC_n$$

$$3V = 2C_2 + 3C_3 + 4C_4 + \dots + nC_n$$

$$6F - 6E + 6V = 12 \Rightarrow 4C_2 + 3C_3 + 2C_4 + C_5 - C_7 - 2C_8 + \dots = 12$$

So at least one of  $C_2, C_3, C_4, C_5$  must be positive.

**The Vitally Important Fact:**

**Every map contains at least one country with 2, 3, 4 or 5 neighbours.**

## 5. Minimal criminals and the six colour theorem

If the six colour theorem is false then of all the counterexamples one or more will contain fewer countries than the rest. These are called minimal criminals.

By the counting formula, a minimal criminal must contain a country bounded by at most five other countries. Show that this is impossible by merging, colouring and reinstating any country with fewer than six neighbours.

## 6. Kempe chains and the five colour theorem.

As above but with 5 colours available, it is easy to prove that a minimal criminal cannot contain a country with 2, 3 or 4 neighbours. Therefore, if the 5CT is false then there must be a minimal criminal containing a country with 5 neighbours and the map cannot be coloured with 5 colours. Proved in 1890 using the method of Kempe chains.

## 7. Method of unavoidable sets, reducible configurations and the four colour theorem

With the 4CT there is a problem when considering minimal criminals which contain a 5-gon. Kempe chains do the job for 2-, 3- and 4-gon but not for a 5-gon.

**A reducible configuration** is any arrangement of countries that cannot occur in a minimal criminal. (so 2-, 3- and 4-gons are reducible configurations as are any set of countries containing at least one of them)

**An unavoidable set** is a collection of countries where any map is bound to contain at least one of them. One example so far is the set given in this handout.

To prove the 4CT amounts to finding an **unavoidable set of reducible configurations**.

Over 1000 hours of computer time checking 100,000 cases. Can this be called a mathematical proof? Heated arguments ensued! Proof or 4CT tidied up in 1994 but still is computer-based, checking all possibilities. Is there an elegant proof out there?

## 8. Related problems.

- In three dimensions?

No number of colours suffice

- The Empire problem. (Heawood 1890)

How many colours are needed if any country may consist of  $r > 1$  distinct portions but no more? (Answer:  $6r$ , proved in 1984)

- Maps on a torus.

Using  $F - E + V = 0$  can show seven colours are sufficient.

- The Heawood conjecture.

For a  $h$ -holed torus Euler's formula is  $F - E + V = 2 - 2h$ . A map on a

$h$ -holed torus requires at most  $\frac{7 + \sqrt{1 + 48h}}{2}$  colours. Proved in 1968.

( $h = 0$  gives four colours but not a proof of 4CT due to assumption  $h > 0$ )

"Four Colours Suffice" by Robin Wilson.  
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