MEI Conference, July 2006

Mechanics 2

Topics that cause problems

General

- There is a need for clear complete diagrams for all of the M2 topics and this will be emphasized in the notes below. Without diagrams, strong candidates make unnecessary errors and weaker candidates often sink in a mire of misunderstanding. There seems to have been a recent increase in the number of students who, apparently, annotate diagrams given on the question paper, leaving the examiner without this essential key to the solution.

- There are many sign errors in expressions. Usually these follow from the lack of definition of the sense of a force or lack of a consistent choice of positive direction (shown in a diagram), but there are other situations commonly seen such as

    - when solving pin-jointed light framework problems, incorrectly carrying forward the *sign* of a force found from one equilibrium equation to a new equation; also, candidates usually do better if they treat all internal forces as tensions as they can then interpret all negative forces as thrusts;

    - inconsistency when deriving work-energy equations with a ‘profit and loss’ method.

- There are two common incorrect uses of friction results.

    - Many candidates learn *Coulomb’s law* as $F = \mu R$ and replace all occurrences of *F* with μR. Having said that friction is limiting it is, of course, impossible to find its value if it is not. Candidates do best with $F \le \mu R$ if they can deal with the inequality but writing $F_{\max} = \mu R$ does also seem effective.

    - Many candidates believe that in all circumstances of an object sliding down a slope inclined at λ, they may use $\mu = \tan\lambda$.

- Candidates frequently reveal an almost total lack of understanding of the significance of their results. This is compounded by a lack of command of technical language so that force, work, energy, impulse etc are often used as elegant variations in a piece of prose.

- Questions frequently ask the candidates to establish a given result in order to ensure that they have the correct expression for further use. The argument must be made **very** clear involving statements such as *resolving vertically, conserving linear momentum in the direction AB*, *taking moments about R* etc.

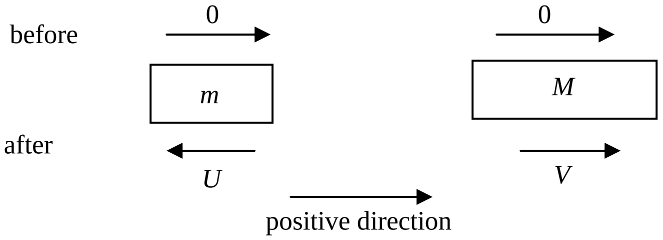
Impulse, linear momentum and impact

- Impulse and momentum are vectors quantities. Problems may be set in more than one dimension, in which case they may be represented by directed line segments. However, even if the motion is in one dimension, clear diagrams are require to establish the directions involved and to establish a positive sense. In particular, calculations of impulse as $m\mathbf{v} - m\mathbf{u}$ often involve sign errors.

- In collisions, by N3L, the forces and hence the impulses in the collisions are of equal magnitude but opposite direction on the two bodies. When no external force acts on a system of particles or bodies that collide, the *total linear momentum of the system is conserved* as the forces involved in the collisions are internal and give zero net impulse. However, the linear momentum of each body involved in a collision *will change* as the force of the collision was external to that body.

- Care must be used by candidates when applying PCLM when relative motion is involved. As an example, suppose that you are firmly seated on a sledge that is on smooth ice and initially at rest. You now throw a ball horizontally away from the sledge. The PCLM tells us that by the time you have the ball moving relative to you, the sledge and you must be moving in the opposite direction to conserve linear momentum; the speed of the ball relative to you will not be the speed of the ball relative to the ground. A particularly important application of this idea is in ballistics; if you fire a gun with a barrel inclined to the horizontal, the recoil will cause it to be moving backwards while the shell is still in it. The true angle of elevation of the shell as it leaves the barrel is *not* the angle of the barrel; this makes ballistics interesting!

Example A woman is sitting firmly on a sledge on smooth, horizontal ice. The sledge is at rest. The combined mass of the woman and the sledge is $M + m$. She throws a ball of mass m horizontally from the sledge with a speed of u relative to her. What is the speed of the sledge when the ball has been thrown?



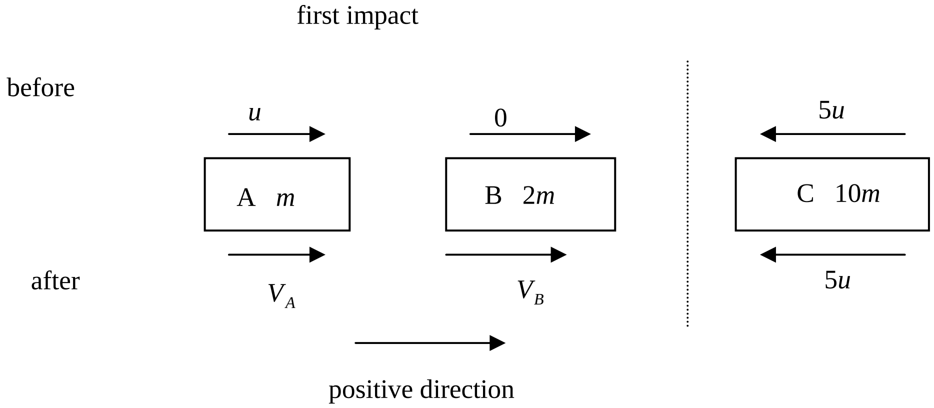
PCLM gives $\longrightarrow$ $MV - mU = 0$

We also have $U + V = u$,

so $MV - m(u - V) = 0$ and $V = \dfrac{mu}{M + m}$.

- Problems involving multiple impacts where the total linear momentum is conserved throughout can often be analysed efficiently by continued reference to the total initial linear momentum.

Example Three particles, A, B and C are in a line and sliding on a smooth horizontal surface. Their masses and initial velocities are shown in the diagram. A collides with B first, then C collides with B and then A, coalescing with them. The coefficient of restitution in the first impact between A and B is 0.5. All the units are SI. Show that A is brought to rest in the first collision and find the velocity of the final, coalesced particle.



For the impact between A and B

PCLM $mu + 0 = mV_A + 2mV_B$

NEL $\dfrac{V_B - V_A}{0 - u} = -0.5$

Solving gives $V_A = 0$.

The next part of the question can be dealt with without considering any further impacts, nor even the one analysed above. Given that the final state is a single coalesced particle, we may use the fact that linear momentum is conserved through each collision.

Let the final velocity be W in the positive direction.

PCLM gives $mu + 0 + 10m \times (-5u) = (m + 2m + 10m)W$ so $W = -\dfrac{49u}{13}$.

- Note that in the example above, NEL was used in the form $\dfrac{v_A - v_B}{u_A - u_B} = -e$.

  In my view, this formulation together with a diagram with a positive sense defined, the sense of each known velocity marked and all unknowns given positive sense, gives fewest sign errors.

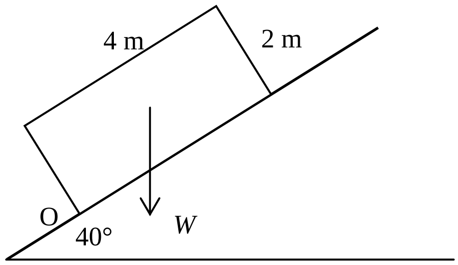
Moments and static equilibrium

- A clear diagram is again vital as missing out forces often completely undermines a solution. A common error is to omit the forces acting at a pivot point (this could be a hinge or a point of contact with another body). This sometimes encourages candidates to suppose that a problem may be solved entirely by resolution; more commonly, the candidate may well start off correctly because moments were taken about the pivot (and the answer confirmed if given in the question) but then go wrong when moments are taken about another point or when resolution is used.

- Some candidates incorrectly apply the analysis of pin-jointed light frameworks to heavy beams.

- In some situations it is not easy to find the perpendicular distance from the axis or a point onto the line of action of the force. In such cases it may be helpful to express the force as the sum of two forces in more convenient directions and then find the total moment of those forces about the axis or point.

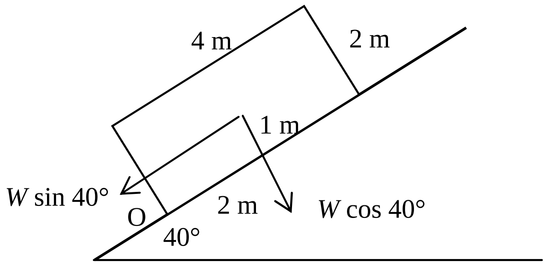
Example The situation shown in the diagram on the right is found in many problems. The weight W acts through the centre of a uniform rectangular body at rest on a slope at 40° to the horizontal. The dimensions of the rectangle are 4 m by 2 m. Calculate the moment of the weight about the point O. Note that, for clarity, no other forces have been shown.



The solution is not too hard if the perpendicular distance from O onto the line of action of W is attempted but the following is much easier.

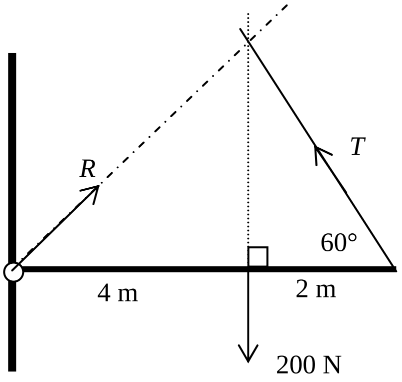
Replace W with its resolved parts parallel to and perpendicular to the plane.

The clockwise moment of W about O is now clearly $W\cos 40 \times 2 - W\sin 40 \times 1$.



- If a body is in equilibrium with several forces acting, the normal reaction will not necessarily have a line of action through the centre of mass and it should not be marked through this point in a diagram. The position of this line of action may be determined by taking moments. Please see the sheet on force diagrams.

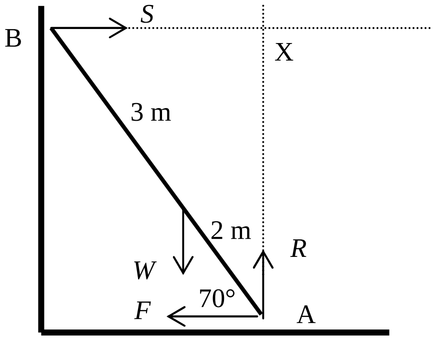
- Three forces acting on a body in equilibrium must be concurrent. This result also gives the possibility of some neat solutions to otherwise quite testing problems. The result is often used in conjunction with a triangle of forces.



Example A bar of length 6 m and weight 200 N is freely hinged at one end A and is kept horizontal by a light string inclined at 60° to the horizontal at the other end B. The centre of mass of the bar is 4 m from A. Find the angle of the reaction at the hinge.

Since there are only three forces, they must be concurrent. The lines of action of the tension in the string and the weight are given by the geometry of the problem. The line of action of the reaction at the hinge, *R*, must pass through the point of intersection of the lines of action of the weight and the tension. It is a matter of simple trigonometry to find the angle of *R*. Note that we did not have to find the magnitudes of *R* or *T*.

- The ability to choose the point about which moments are taken can also lead to some neat solutions.



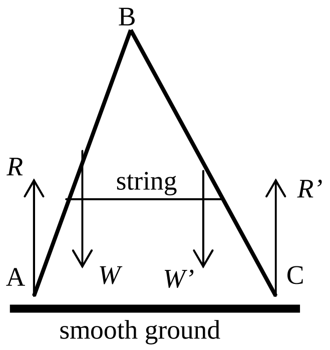
Example The diagram shows a ladder in equilibrium resting on a smooth vertical wall and on rough horizontal ground. Suppose we want to find the frictional force, *F*, acting on the foot of the ladder.

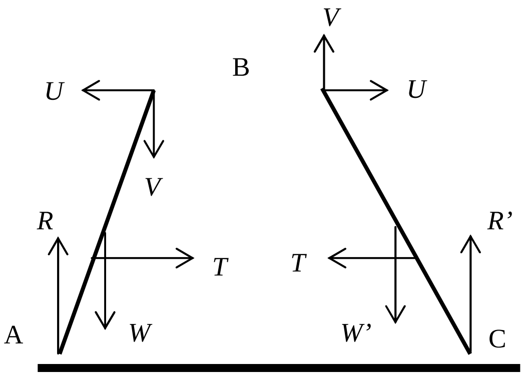
We could take moments about A and resolve horizontally and vertically but a neater method is available.

X is the intersection of the lines of action of *R* and *S*, neither of which we need to know to find *F*.

Taking clockwise moments about X gives $F \times 5\sin 70 - W \times 2\cos 70 = 0$ and the result follows.

- In problems involving stepladders, it is clear that if the stepladder is treated as a single body, then applying the condition for equilibrium cannot let us determine the tension in the string holding the legs together nor find the force in the hinge, as these will be internal forces. It is usually easiest first to consider the equilibrium of the whole stepladder but then the condition for equilibrium must be applied to each leg. In a case where the legs have equal length, weight and weight distribution, there are arguments from symmetry that can usefully be applied. A person standing on the ladder will destroy this symmetry.





- When it is necessary to use the total moment of the weight of a composite body, it is often better first to find the centre of mass of that body so that calculations do not have to be repeated.

Work, energy and power

- The *work done* (WD) by a constant force **F** *on* a body which is displaced by **s** is $\mathbf{F}\bullet\mathbf{s}$, or $Fs\cos\theta$, where θ is the angle between the vectors **F** and **s**. This can be viewed as the product of the distance moved with the component of the force in the direction of the displacement or as the product of the magnitude of the force with the component of the displacement in the direction of the force. Care should be used to ensure that if the *resolved parts* of F or s are used in place of one of them that the WD is calculated as Fs not as $Fs\cos\theta$.

- When using the work-energy equation, care must be taken to ensure that the signs of the terms are correct. Energy may be gained or lost by a body according to the direction in which it is moving. For example, a particle falling under gravity will gain kinetic energy but one thrown up and rising under gravity will lose kinetic energy. Work done against friction always reduces the energy of the body as the frictional force is in the opposite direction to the motion.

- When a number of forces act on a body we may apply the work-energy principle so long as the total work done by all of the forces is calculated. There is no need separately to consider any PE changes, these will have been included as the work done against the conservative forces.

We may alternatively consider the conservative forces as leading to conservation and then add in the energy change due to the non-conservative forces.

In either case, common mistakes by students are

(usually in the absence of a force diagram), to neglect the work done by one or more of the forces,

to calculate the work done by *all* of the forces and then add to this any PE change, thereby including this energy term twice.

- Many students try to apply N2L to problems where an energy approach is preferable or even essential. The energy approach typically does not require the assumptions that forces are constant that are necessary for the direct application of Newton' second law and the constant acceleration results; saying that 40 J of work is done against resistance when an object is moved 5 m does not mean that that the force is a constant 8 N.

Example A small heavy object is swinging as a pendulum at the end of a light inextensible string above horizontal ground. There is no air resistance. If the string breaks, the object will hit the ground at the same speed, regardless of where in its motion the string breaks.

The only force is that due to gravity so the force is conservative.

Hence KE + PE = constant.

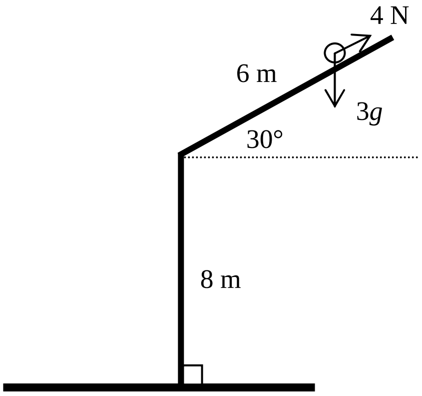
Suppose the object has mass m and is h above the ground when it is stationary at the end of a swing. The PE of the system is mgh relative to the ground.

Hence, at any time, PE + KE = mgh.

But, when the object hits the ground its PE is zero, relative to the ground. So the object hits the ground with KE of mgh wherever the string breaks.

Incidentally, equating $\frac{1}{2}mv^2 = mgh$ will give this speed.

Example A small brick of mass 3 kg slides from rest down the roof of a shed against a constant resistance of 4 N and then falls to the ground under gravity in the absence of air resistance. The dimensions are given on the diagram; the brick slides 6 m down the roof. At what speed does the brick hit the ground?



This problem is very hard if one uses N2L, especially if the scalar product is not known. The energy approach is easy, especially if the conservative and dissipative forces are considered separately.

Relative to the ground, the GPE is $3g(8+6\sin 30)$ J = $33g$ J

The work done against friction is $6\times 4 = 24$ J.

The initial speed is zero and the final speed is v, say. There is no work done against air resistance.

By the work-energy principle

$\frac{1}{2}\times 3\times v^2 = 33g - 24$, hence v = 14.1 m s^{-1} (3 s.f.)

It is worth noting the approach taken at the end of this problem, where, effectively, the work done on the system was calculated as the GPE lost less the work done against friction. Many students find this sort of approach natural but can make mistakes with signs. The work-energy principle may be applied by first considering the work done by all the forces acting **down** the plane until the bottom of the roof is reached. This is clearly

$$(3g\sin 30 - 4)\times 6 \;=\; 9g - 24 \text{ J.}$$

Subsequently considering the brick as a projectile moving against negligible resistance gives total work done of

$$9g - 24 + 3g\times 8 = 33g - 24 \text{ at ground level.}$$

Note that in this method we do not separately consider the GPE change; this is simply a special name given to one of the terms in the expression.

Centres of mass

- When attempting to find the centre of mass of a body, students who use vector methods are usually more successful than those who attempt an approach using moments, especially when three coordinates are involved.

  In my view, success at this level comes directly from a clear understanding of the *additive principle.*

  if one system of particles, S, has total mass M_{S} and centre of mass $\mathbf{r}_{\mathrm{S}}$ relative to an origin O and another system of particles, T, has total mass M_{T} and centre of mass $\mathbf{r}_{\mathrm{T}}$ relative to O, then the combined system of particles $S + T$ has position vector relative to O, $\mathbf{r}_{\mathrm{G}}$, given by

  $$(M_{\mathrm{S}} + M_{\mathrm{T}})\mathbf{r}_{\mathrm{G}} = M_{\mathrm{S}}\,\mathbf{r}_{\mathrm{S}} + M_{\mathrm{T}}\,\mathbf{r}_{\mathrm{T}}$$

  (i.e. it is as if each of the two systems may be replaced by a single particle with the mass of that system positioned at the centre of mass of that system).

- The additive principle may be extended to any number of systems, making it easy to find the position of the centre of mass of any composite body where the component parts themselves have centres of mass in positions that may be determined easily.

- Students are often too ready to throw away information already determined about the positions of centres of mass. When a composite body gains or loses parts it is rarely better to start from scratch and usually better to use the additive principle.

- Although solutions are generally best set out in full vector form, we often know one or more of the coordinates of the centre of mass of a body by arguments from symmetry. The remaining coordinates are then often best dealt with separately by means of scalar equations.

- It is customary to ‘look up’ and quote the positions of centres of masses for common laminae and solids. A useful result is that the c.m. of a uniform lamina in the shape of a triangle is at the point of intersection of the medians (the centroid of the triangle).

- When dealing with laminae and with lines, it is not convenient to use the actual density of the materials but instead to use *surface density* for thin sheets and *line density* for thin rods. The usual letters used for these are σ and λ ; their units are mass per unit area and mass per unit length respectively.

- Many problems involve freely hanging a lamina or a body so that it then rests in equilibrium. The condition for this is that G should be directly below the point of suspension.

- Arguments based on moments are often neater and more direct for solving some problems involving centres of mass, for instance, ones involving the angle of a suspended body.

- A further application of the additive principle is to some problems where part of a body or a lamina has been removed.

Example The uniform lamina OABCDEFG has the dimensions in metres shown in Fig.1.

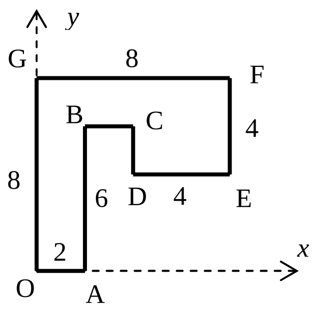


Fig.1

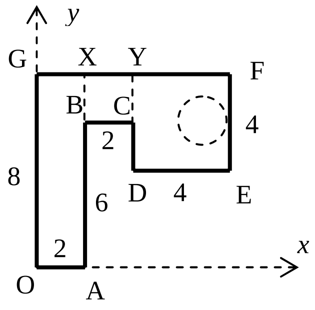


Fig.2

(i) Calculate the coordinates of the centre of mass of the lamina, referred to the axes shown in Fig.1.

(ii) A circular disc of radius 1 m and centre (7, 6) is cut from the lamina, as shown in Fig.2. Calculate the coordinates of the new position of the centre of mass.

For part (i), we must divide up the figure. One way is shown in Fig.2, giving

the rectangle OAXG	area 16	c.m. (1, 4)
the rectangle BCXY	area 4	c.m. (3, 7)
the rectangle DEFY	area 16	c.m. (6, 6)

This gives $36\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 16\begin{pmatrix}1\\4\end{pmatrix} + 4\begin{pmatrix}3\\7\end{pmatrix} + 16\begin{pmatrix}6\\6\end{pmatrix} = \begin{pmatrix}124\\188\end{pmatrix}$

So $\bar{x} = \dfrac{31}{9}$ and $\bar{y} = \dfrac{47}{9}$; (these are 3.44 and 5.22 to 3 s. f.)

Students should always show how they have divided up such a shape and not just mark the diagram on the question paper.

For part (ii), we cannot easily build the shape up directly as we do not know the c.m. of the section DEFY with the circle removed. The best way is to argue that the required part with the circular disc added gives the answer to part (ii).

We now have

the circular disc	area $\pi \times 1^2$	c.m. (7, 6)
the total figure with disc present	figures from part (i)	
the required shape	area $(36 - \pi)$	c.m. $(\bar{x}, \bar{y})$

This gives $\pi\begin{pmatrix}7\\6\end{pmatrix} + (36 - \pi)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}124\\188\end{pmatrix}$ so $\bar{x} = 3.10$ and $\bar{y} = 5.15$ to 3 s. f.

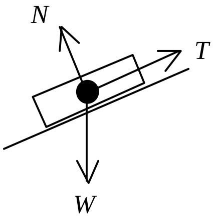
Note that we could divide *this* figure into parts that are symmetrical and use the additive principle directly. However, if a disc or a square is removed asymmetrically from, say, a circular disc then it is not possible to use the additive principle directly and we have to adopt the strategy used in the example above.

Drawing and labelling diagrams in mechanics

Clear, accurate diagrams are an essential step in the correct analysis of a variety of mechanics problems. They provide a means of recording the information given, make it possible to define an origin and a positive direction and frequently reveal aspects of the situation that suggest efficient methods of solution. Solutions attempted without diagrams frequently lead to inconsistencies in the analysis. Poor or wrong diagrams can easily lead to false assumptions being made.

Force diagrams

1 Decide on a particle or a body model according to whether any moments are involved.



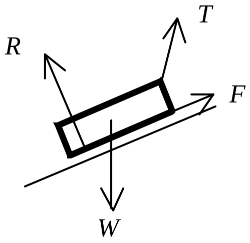
If no moments are to be considered a particle model is always adequate; this may involve

a single particle (e.g. representing a car),

several particles (e.g. representing a car and caravan connected by a tow-bar).

The particle may be placed inside a representation of the object if this aids clarity.

You may always use this model in MEI M1.



If moments are involved, use a body model and consider the lines of action of the forces. Note that *R* has not been marked as passing through the centre of mass.

It often helps clarity slightly to separate items, e.g. the block and the slope.

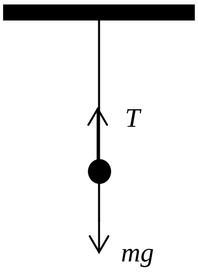
This model *may* be necessary in MEI M*n*, *n* >1, but often a particle model is adequate.

2 Follow any instructions given in a question. You may be asked to mark in all the forces or only those with components in a particular direction. If in doubt mark in all the forces acting.

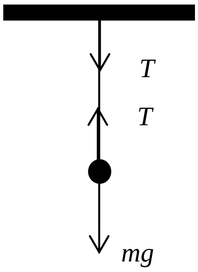
The following are subject to point 2.

3 The diagram should represent all the forces acting at a point, or on a body or system. Identify that point, body or system and do *not* include any forces not acting on it.

Example A particle of mass *m* is attached to one end of a light string. The other end of the string is fixed to a ceiling. The particle is in equilibrium. Mark in all the forces acting on the particle.



The forces acting on the particle hanging from a ceiling are shown on the left. The extra force shown on the right does not act on the particle and should not be included.

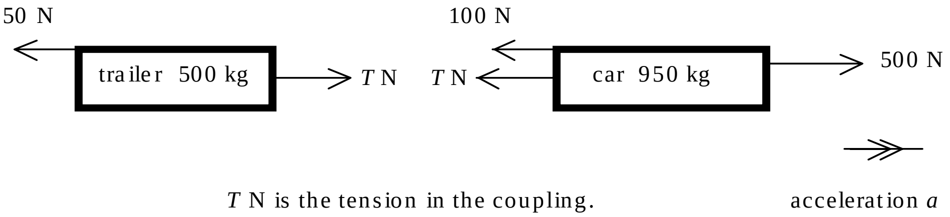


It is often better *not* to indicate an internal force pair on a single diagram. If you have to consider separately the forces acting on two or more bodies connected or in contact, it is often clearer to use linked diagrams instead of a single diagram. Clearly label the pairs of forces given by N3L with equal magnitudes and opposite directions.

4 Put in any forces given in the question, arrowed in the direction of the force and with any given magnitudes marked. Indicate any angles given.

5 In dynamics problems indicate the direction(s) you are taking to be positive.

Example A car of mass 950 kg and with a driving force of 500 N is pulling a trailer of mass 500 kg. The coupling between the car and the trailer is light, horizontal and rigid. The resistances to the motion of the car and the trailer are 100 N and 50 N respectively. Draw a diagram showing the forces acting on the car and trailer in the direction of motion, including the internal force in the coupling.



T N is the tension in the coupling.

acceleration a

Note that this is effectively a particle model. The forces are separated for clarity.

6 If any of the particles or bodies has a mass, mark and label the corresponding weight. In a body model the weight must act through the centre of mass.

7 If any *surfaces* are in contact,

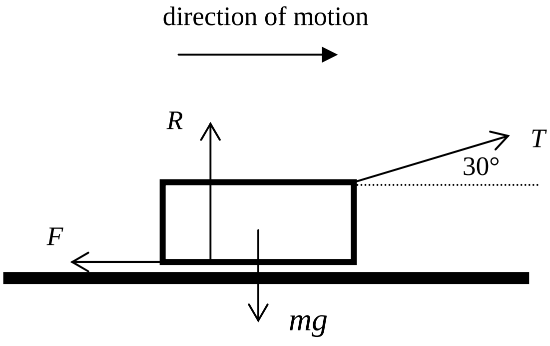
(a) mark in the normal reaction at 90° to the tangent of contact,

(b) determine whether or not the surfaces are smooth; if *not*, indicate a friction force along the tangent of contact in the direction *opposite to the movement of the body (or opposite to the direction the movement would take place if there were no friction).*

NB1 The friction force opposes movement (i.e. velocity) **not** acceleration.

NB2 If the direction of the friction force cannot be determined produce diagrams for both cases.

Example A block of mass m is pulled along a horizontal plane by a string at 30° to the horizontal.



8 If any strings are involved, mark in the force in the line of the string as a tension.

9 If there are any hinges and resolution is required, it is usually better to mark in the reaction force as *two components at right angles* instead of a single force at an unknown angle. However, the reaction at the hinge is better left as a single force if this reduces the problem to one of only three forces, for which special techniques may be employed.

General points to check.

10 Ensure all forces have labels and arrows; if the direction of a force is known, mark it as being in that direction. Do use the same label for any forces with known equal magnitudes. Do *not* use the same label for forces not known to have the same magnitude. Apply this rule to linked diagrams as well as single diagrams.

11 Make sure that you have not made any unwarranted assumptions about magnitudes and directions.

In particular

Never mark in friction as taking its limiting value; make this statement in your working only after checking that it is true. In general $F \le F_{\max} = \mu R$.

When dealing with a normal reaction in a body model, consider carefully its line of action; it may well not pass through the centre of mass.

12 For a *statics problem*, having completed your force diagram, consider whether a force polygon will help your analysis.

For a *three force statics problem*, remember that the forces must be concurrent and consideration of this fact as well as the triangle of forces will often provide a simple method of solution.

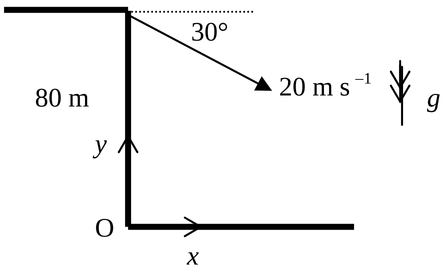
13 We often want to show forces, velocities and accelerations on the same diagram. The best way to do this without leading to confusion is to use different sorts of arrows. There is no standard convention but I have used

single arrows for forces	→
double arrows for acceleration	↠
blocked arrows for velocity	⭢

Diagrams for kinematics problems

These are often omitted and much confusion follows. It is always necessary to establish the direction being taken as positive and often necessary to note the position of the origin. A particularly important application is to projectile motion where the sense of positive and the position of the origin may well not be clear from the context.

Example A small stone is projected downwards at 30° to the horizontal from the top of a vertical cliff that is 80 m above plane horizontal ground. Its initial speed is 20 m s^{-1}.



Note that with the convention of the diagram, the initial value of y is 80, the initial vertical component of velocity is negative, g is in the negative direction and that the stone hits the sea when $y = 0$.

Suppose that the origin were taken at the top of the cliff, still with y upwards. Now the initial value of y is zero and the stone hits the sea when $y = -80$.

Suppose now that the positive value of y is downwards with the origin at the top of the cliff. The initial value of y is zero, the initial vertical component of velocity is positive, g is in the positive direction and the stone hits the sea when $y = 80$.

Suppose that the positive value of y is downwards with the origin at the bottom of the cliff.........

A clear diagram defining the origin and positive directions is vital!

Diagrams for impulse, momentum and impact problems

We are dealing with vector quantities and so once again it is vital that the sense of each such quantity clear. It is also vital that the direction taken to be positive should be clearly marked.

Example Two equal spheres A and B, each of mass m are on a smooth horizontal table. Initially, sphere A is at rest and sphere B is travelling directly towards it with a speed u. The coefficient of restitution in the subsequent collision is e. Find the impulse acting on A in the collision.