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Newton, Euler and the binomial theorem

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Newton's approximation to π

This is probably most effective if students use the second version two pages on from here and answer the questions. The complete version with answers starts here…

In around 225BC Archimedes deduced that $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

The next significant breakthrough in the attempt to pin down π was made almost 1900 years later by Isaac Newton.

First consider the circle, centre $\left(\tfrac{1}{2},0\right)$ radius $\tfrac{1}{2}$.

This has equation $(x-\tfrac{1}{2})^2 + y^2 = (\tfrac{1}{2})^2$ which simplifies to $y^2 = x(1-x)$ or $y = x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$

By the binomial theorem this is

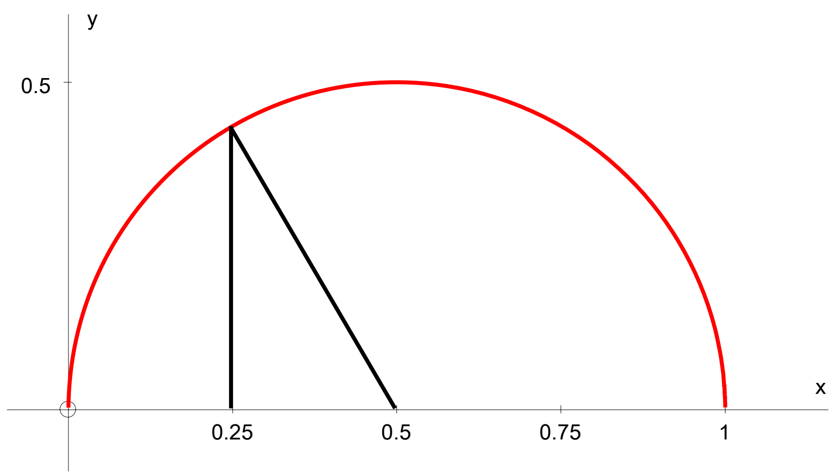
$$y = x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} = x^{\frac{1}{2}}\left(1+\frac{1}{2}(-x)+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)(-x)^2}{2!}+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(-x)^3}{3!}+\ldots\right)$$

which simplifies to

$$y = x^{\frac{1}{2}}\left(1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\frac{5}{128}x^4-\frac{7}{256}x^5-\frac{21}{1024}x^6-\frac{33}{2048}x^7-\ldots\right)$$

$$= x^{\frac{1}{2}}-\frac{1}{2}x^{\frac{3}{2}}-\frac{1}{8}x^{\frac{5}{2}}-\frac{1}{16}x^{\frac{7}{2}}-\frac{5}{128}x^{\frac{9}{2}}-\frac{7}{256}x^{\frac{11}{2}}-\frac{21}{1024}x^{\frac{13}{2}}-\frac{33}{2048}x^{\frac{15}{2}}-\ldots \qquad (\dagger)$$

Now consider the area contained by the x-axis, the circle above the x-axis and the line $x=\frac{1}{4}$:



The angle subtended at the centre of the circle is 60° (since the triangle is half an equilateral triangle) and so the required area is

$$\frac{\pi\left(\frac{1}{2}\right)^2}{6}-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{\sqrt{3}}{4}=\frac{\pi}{24}-\frac{\sqrt{3}}{32}$$

But a close approximation to this area can also be found by integrating $(\dagger)$ between the limits $x=0$ and $x=\frac{1}{4}$:

$$=\int_0^{1/4} x^{1/2}-\frac{1}{2}x^{3/2}-\frac{1}{8}x^{5/2}-\frac{1}{16}x^{7/2}-\frac{5}{128}x^{9/2}-\frac{7}{256}x^{11/2}-\frac{21}{1024}x^{13/2}-\frac{33}{2048}x^{15/2}-\ldots\mathrm{d}x$$

$$=\left[\frac{2}{3}x^{3/2}-\frac{1}{5}x^{5/2}-\frac{1}{28}x^{7/2}-\frac{1}{72}x^{9/2}-\frac{5}{704}x^{11/2}-\frac{7}{1664}x^{13/2}-\frac{7}{2560}x^{15/2}-\frac{33}{17408}x^{17/2}-\ldots\right]_0^{1/4}$$

$$=\frac{2}{3}\cdot\left(\frac{1}{2}\right)^3-\frac{1}{5}\cdot\left(\frac{1}{2}\right)^5-\frac{1}{28}\cdot\left(\frac{1}{2}\right)^7-\frac{1}{72}\cdot\left(\frac{1}{2}\right)^9-\frac{5}{704}\cdot\left(\frac{1}{2}\right)^{11}-\frac{7}{1664}\cdot\left(\frac{1}{2}\right)^{13}-\frac{7}{2560}\cdot\left(\frac{1}{2}\right)^{15}$$

$$=\frac{1}{12}-\frac{1}{160}-\frac{1}{3584}-\frac{1}{36864}-\frac{5}{1441792}-\frac{7}{13631488}-\frac{7}{83886080}-\frac{33}{2281701376}-\ldots$$

$$\simeq 0.076773109$$

Comparing this with the exact answer gives

$$\frac{\pi}{24}-\frac{\sqrt{3}}{32}\simeq 0.076773109 \qquad\Rightarrow\qquad \pi\simeq 24\left(\frac{\sqrt{3}}{32}+0.076773109\right)\simeq 3.141592731....$$

So with only eight terms of the infinite series we have found a value of π which turns out to be accurate to seven decimal places!

Newton actually calculated the first 20 terms of the binomial expansion (as well as $\sqrt{3}$ to a similar degree of accuracy) giving π to 16 decimal places. A remarkable calculation but one which caused Newton to comment

"I am ashamed to tell you to how many places of figures I carried these computations, having no other business at the time."

Newton's approximation to π

In around 225BC Archimedes deduced that $3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}$.

The next significant breakthrough in the attempt to pin down π was made almost 1900 years later by Isaac Newton.

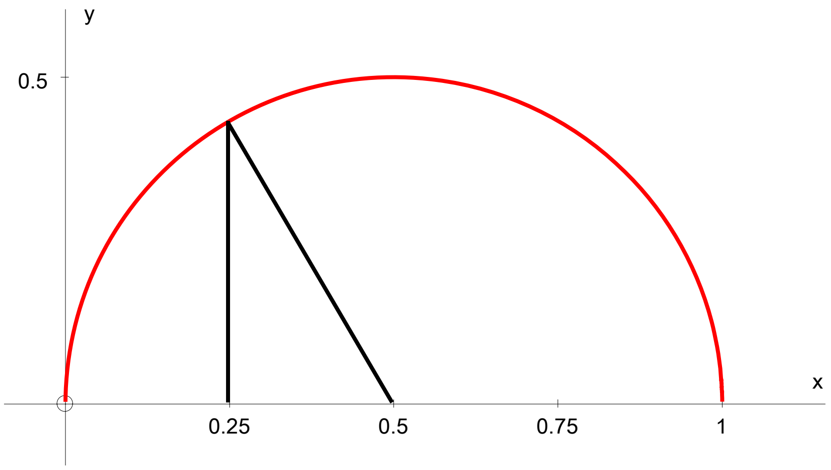
First consider the circle, centre $(^1/_2,0)$ radius $^1/_2$.

1. Show that the semicircle lying in the first quadrant has equation $y = x^{1/2}(1-x)^{1/2}$

2. By using the binomial theorem show that the first five terms agree with the first five terms in the following expression

$$y = x^{1/2} - \frac{1}{2}x^{3/2} - \frac{1}{8}x^{5/2} - \frac{1}{16}x^{7/2} - \frac{5}{128}x^{9/2} - \frac{7}{256}x^{11/2} - \frac{21}{1024}x^{13/2} - \frac{33}{2048}x^{15/2} - \dots \qquad (\dagger)$$

Now consider the area contained by the *x*-axis, this semi-circle and the line $x = \dfrac{1}{4}$:



3. What is the angle subtended at the centre of the circle by the radius shown?

4. Show that the area required is $\dfrac{\pi}{24} - \dfrac{\sqrt{3}}{32}$

5. Explain how to use $(\dagger)$ (and integration) to find an approximation to this area.

6. Using all the terms in $(\dagger)$ show that this approximation is

$$\frac{1}{12} - \frac{1}{160} - \frac{1}{3584} - \frac{1}{36864} - \frac{5}{1441792} - \frac{7}{13631488} - \frac{7}{83886080} - \frac{33}{2281701376} - \dots \simeq 0.076773109$$

7. Compare this with the exact answer found in 4 above. To what level of accuracy does this give the value of π ?

Partitions

Define O(*n*) to be the number of ways of expressing the integer *n* as the sum of odd integers.

For example $O(6) = 4$, the four ways being 5+1, 3+3, 3+1+1+1, 1+1+1+1+1+1

1. Write down all six ways of writing 8 as the sum of odd integers.

Define D(*n*) to be the number of ways of expressing the integer *n* as the sum of distinct positive integers.

For example $D(6) = 4$, the four ways being 6, 5+1, 4+2, 3+2+1

2. Write down all eight ways of writing 9 as the sum of distinct positive integers.

3. Work out and fill in the values of $O(n)$ and $D(n)$ in the table below:

n	1	2	3	4	5	6	7	8	9	10
$O(n)$						4		6		
$D(n)$						4			8	

For any value of n you should find that $\mathrm{O}(n) = \mathrm{D}(n)$.

How can we prove this?

It took the genius and insight of Leonhard Euler to devise the following proof.

First of all, look at the infinite polynomial

$$Q(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)...$$

Think about the coefficient of x^6 when these brackets are multiplied out.

x^6 can arise in exactly four ways:

$x^1.x^2.x^3.1.1.1.1... \quad + \quad x^1.1.1.1.x^5.1.1... \quad + \quad 1.x^2.1.x^4.1.1.1.... \quad + \quad 1.1.1.1.1.x^6.1....$

and these four ways correspond exactly to the four ways of writing 6 as the sum of distinct positive integers (as in $\mathrm{D}(6)$):

1+2+3 1+5 2+4 6

4. Write down the different ways of getting x^7 from the polynomial $Q(x)$.

Explain in your own words why it is no coincidence that the number of ways it can be done is the same as the value of $\mathrm{D}(7)$ in your table

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It can be seen (and checking other coefficients confirms this) that

$$Q(x) = 1 + \mathrm{D}(1)x + \mathrm{D}(2)x^2 + \mathrm{D}(3)x^3 + \mathrm{D}(4)x^4 + ...$$

Now think about the infinite polynomial $\mathrm{P}(x)=\frac{1}{1-x}.\frac{1}{1-x^3}.\frac{1}{1-x^5}.\frac{1}{1-x^7}.\frac{1}{1-x^9}...$

Expand each of these algebraic fractions using the binomial theorem:

$$\mathrm{P}(x)=\left(1+x^1+x^{1+1}+x^{1+1+1}+...\right)\left(1+x^3+x^{3+3}+x^{3+3+3}+...\right)\left(1+x^5+x^{5+5}+x^{5+5+5}+...\right)...$$

Now think about the coefficient of x^6 when these brackets are multiplied out.

x^6 can arise in exactly four ways:

$$x^{1+1+1+1+1+1}.1.1.1... \quad + \quad x^{1+1+1}.x^3.1.1... \quad + \quad x^1.1.x^5.1.... \quad + \quad 1.x^{3+3}.1.1...$$

and these four ways correspond exactly to the four ways of writing 6 as the sum of odd integers (as in $\mathrm{O}(6)$):

1+1+1+1+1+1   3+1+1+1   5+1   3+3

5. Write down the different ways of getting x^7 from the polynomial $\mathrm{P}(x)$.

Explain in your own words why it is no coincidence that the number of ways it could be done is the same as the value of $\mathrm{O}(7)$ in your table.

It can be seen (and checking other coefficients confirms this) that

$$\mathrm{P}(x)=1+\mathrm{O}(1)x+\mathrm{O}(2)x^2+\mathrm{O}(3)x^3+\mathrm{O}(4)x^4+...$$

So, **if** (and it's a big if) we could show that $\mathrm{P}(x) \equiv \mathrm{Q}(x)$ then we would have proved that $\mathrm{O}(n) = \mathrm{D}(n)$ for all values of *n*.

The problem is they don't look the same:

$$\mathrm{P}(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^7} \cdot \frac{1}{1-x^9} \cdots$$

$$\mathrm{Q}(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots$$

Nevertheless, start with

$$\mathrm{Q}(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\ldots$$

Introduce a few 1s:

$$\mathrm{Q}(x) = (1+x) \times 1 \times (1+x^2) \times 1 \times (1+x^3) \times 1 \times (1+x^4) \times 1 \times (1+x^5)\ldots$$

Write these 1s in the form $\frac{(1-x)}{(1-x)}$, $\frac{(1-x^2)}{(1-x^2)}$, etc:

$$\mathrm{Q}(x) = (1+x)\frac{(1-x)}{(1-x)}(1+x^2)\frac{(1-x^2)}{(1-x^2)}(1+x^3)\frac{(1-x^3)}{(1-x^3)}(1+x^4)\frac{(1-x^4)}{(1-x^4)}(1+x^5)\frac{(1-x^5)}{(1-x^5)}..$$

and now look to see what cancels:

$$\mathrm{Q}(x) = \cancel{(1+x)}\frac{\cancel{(1-x)}}{(1-x)}\cancel{(1+x^2)}\frac{\cancel{(1-x^2)}}{\cancel{(1-x^2)}}(1+x^3)\frac{(1-x^3)}{(1-x^3)}(1+x^4)\frac{(1-x^4)}{\cancel{(1-x^4)}}(1+x^5)\frac{(1-x^5)}{(1-x^5)}..$$

In fact all the numerators will cancel leaving only

$$\mathrm{Q}(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^7} \cdot \frac{1}{1-x^9} \cdots$$

and this is identical to $\mathrm{P}(x)$!

Therefore we have proved (or Euler did in 1740) that $\mathrm{O}(n) = \mathrm{D}(n)$ for all values of *n*.

"Read Euler, read Euler. He is the master of us all."
- *Laplace*

Alternative proof

Any number can be expressed uniquely in the binary scale:

$$l = 2^a + 2^b + 2^c + \ldots \quad (0 \le a < b < c\ldots)$$

Hence a partition of *n* into odd parts can be written as

$$\begin{aligned} n &= l_1.1 + l_2.3 + l_3.5 + \ldots \\ &= \left(2^{a_1} + 2^{b_1} + 2^{c_1} + \ldots\right)1 + \left(2^{a_2} + 2^{b_2} + 2^{c_2} + \ldots\right)3 + \left(2^{a_3} + 2^{b_3} + 2^{c_3} + \ldots\right)5 + \ldots \end{aligned}$$

and there is a one-to-one correspondence between this partition and the partition into unequal parts:

$$2^{a_1}, 2^{b_1}, 2^{c_1}, \ldots, 2^{a_2} \times 3, 2^{b_2} \times 3, 2^{c_2} \times 3, \ldots, 2^{a_3} \times 5, 2^{b_3} \times 5, 2^{c_3} \times 5, \ldots$$

Example.

The partition $1+3+3+3+3+3+5+7+7+7$ of 42 into odd parts corresponds to

$$\begin{aligned} (1)1+(5)3+(1)5+(3)7 &= (1)1+\left(2^2+1\right)3+(1)5+\left(2^1+1\right)7 \\ &= 1+\left(2^2.3+1.3\right)+5+(2.7+7) \\ &= 1+(12+3)+5+(14+7) \end{aligned}$$

which is a partition of 42 into distinct parts.

Partitions of 11:

Odd partition	11	9	7	5	3	1	Distinct partition
11	1						11
9+1+1		1				2	9+2
7+3+1			1		1	1	7+3+1
7+1+1+1+1			1			4	7+4
5+5+1				2		1	10+1
5+3+3				1	2		5+6
5+3+1+1+1				1	1	3 = 2+1	5+3+2+1
5+1+1+1+1+1+1				1		6 = 4+2	5+4+2
3+3+3+1+1					3 = 2+1	2	6+3+2
3+3+1+1+1+1+1					2	5 = 4+1	6+4+1
3+1+1+1+1+1+1+1+1					1	8	3+8
1+1+1+1+1+1+1+1+1+1+1						11 = 8+2+1	8+2+1