MEI

Mathematics in Education and Industry

MEI STRUCTURED MATHEMATICS

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How Tartaglia can help to mark C3 coursework.

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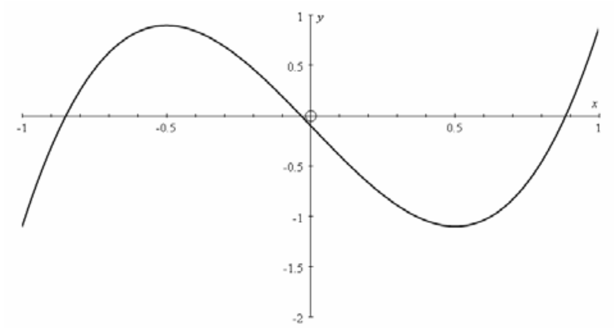
The solution of (some) cubic equations

1. Vieta

It was Vieta who first saw the connection between the triple angle formula and the solution of some cubic equations.

$\text{Cos}3\theta = 4\cos^3\theta - 3\cos\theta$

So consider the cubic equation $4x^3 - 3x = 0.1$



Substituting $x = \cos\theta \Rightarrow 4\cos^3\theta - 3\cos\theta = 0.1$

$\Rightarrow \cos 3\theta = 0.1 \Rightarrow 3\theta = \arccos 0.1 = 1.4706$

$\Rightarrow \theta = 0.4902 \Rightarrow x = 0.8822$

But also $\theta = \arccos 0.1 = 2\pi - 1.4706 = 4.813$

$\Rightarrow \theta = 0.1.6042 \Rightarrow x = -0.0334$

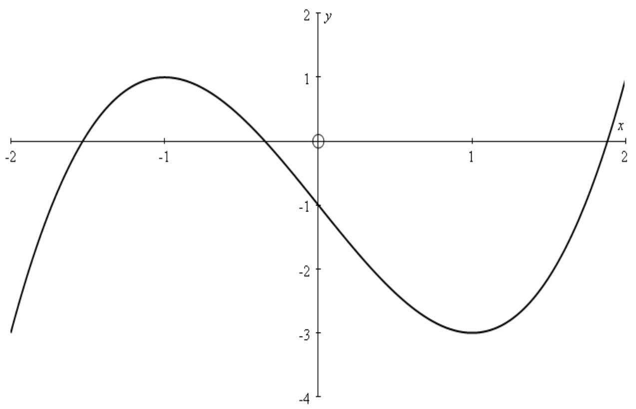
But also $\theta = \arccos 0.1 = 2\pi + 1.4706 = 7.7538$

$\Rightarrow \theta = 2.5846 \Rightarrow x = -0.8489$

The exact solution is $x = \cos\left(\frac{1}{3}\arccos 0.1\right)$

Equations can be transformed.

Consider the cubic equation $x^3 - 3x = 1$



Substituting $x = kz$ gives $(kz)^3 - 3kz = 1$

$\Rightarrow z^3 - \frac{3}{k^2}z = \frac{1}{k^3} \Rightarrow 4z^3 - \frac{12}{k^2}z = \frac{4}{k^3}$

So put $\frac{12}{k^2} = 3 \Rightarrow k = 2,$ giving $4z^3 - 3z = 0.5$

Now substitute $z = \cos\theta \Rightarrow 4\cos^3\theta - 3\cos\theta = 0.5$

$\Rightarrow \cos 3\theta = 0.5 \Rightarrow 3\theta = \arccos 0.5 = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\Rightarrow x = 2\cos\frac{\pi}{9}, 2\cos\frac{5\pi}{9}, 2\cos\frac{7\pi}{9}$

i.e. $x \approx 1.8794, -0.3473, -0.3830$

The solution to the general equation

$x^3 - mx = n$

is $x = \sqrt{\frac{4m}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3n}{2}\sqrt{\frac{3}{m^3}}\right)\right)$

Does the method break down?

YES!

When $\left|\left(\frac{3n}{2}\sqrt{\frac{3}{m^3}}\right)\right| > 1$

i.e. when $27n^2 > 4m^3$

For example take a case at the limit; $m = 3$ and $n = 2$ giving the equation $x^3 - 3x = 2$

$$x = \sqrt{\frac{4m}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3n}{2}\sqrt{\frac{3}{m^3}}\right)\right) = \sqrt{4}\cos\left(\frac{1}{3}\arccos\left(3\sqrt{\frac{1}{9}}\right)\right) = 2\cos\left(\frac{1}{3}\arccos 1\right)$$

$x = 2\cos\left(\frac{1}{3}\times 2r\pi\right)$ for $r = 0, 1, 2....$

$x = 2, -1, -1$

Example: Solve the equation $x^3 - 2x = 1$

2. Cardan

Consider the cubic equation $x^3 + mx = n$

Now note that $(a - b)^3 = a^3 - 3a^3b + 3\,ab^2 - b^3 = (a^3 - b^3) - 3ab(a - b)$
i.e. $(a - b)^3 + 3ab(a - b) = (a^3 - b^3)$

Substituting $x = a - b$ gives $x^3 + mx = n$
where $3ab = m$ (i)
$(a^3 - b^3) = n$ (ii)

From (i), $b = \dfrac{m}{3a}$ in (ii) gives $a^3 - \dfrac{m^3}{27a^3} = n \Rightarrow \left(a^3\right)^2 - n\left(a^3\right) - \dfrac{m^3}{27} = 0$

This is a quadratic equation in a^3.

Solve to find a, then in (i) find b.

Then $x = a - b$.

E.g. Solve the equation $x^3 - 3x = 7$.

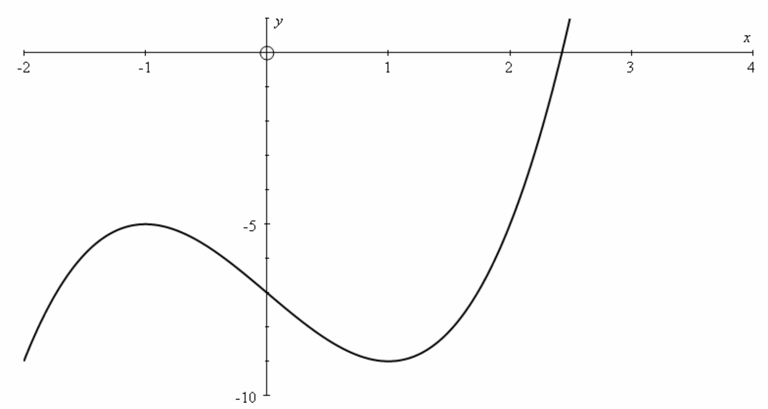
$$3ab = -3 \Rightarrow b = -\frac{1}{a}$$

$$\Rightarrow a^3 + \frac{1}{a^3} = 7 \Rightarrow \left(a^3\right)^2 - 7\left(a^3\right) + 1 = 0$$

$$\Rightarrow a^3 = \frac{7 \pm \sqrt{45}}{2} \approx 6.854$$

$$\Rightarrow a \approx 1.899 \Rightarrow b = -\frac{1}{1.899} = -0.5264$$

$$\Rightarrow x = 2.4254$$



The exact solution is $x = \sqrt[3]{\dfrac{7+\sqrt{45}}{2}} + \sqrt[3]{\dfrac{2}{7+\sqrt{45}}} = \sqrt[3]{\dfrac{7+\sqrt{45}}{2}} + \sqrt[3]{\dfrac{7-\sqrt{45}}{2}}$

Does this method break down?

YES!!
Quadratics do not always have real roots.

3. Can all cubics of this form be solved by one of these methods?

E.g. $x^3 - 3x = 7$ was solved by Cardan. It has $m = 3$ and $n = 7$ and so $27n^2 > 4m^3$

So the equation can be solved by Cardan but not by Vieta.

$x^3 - 3x = 1$ was solved by Vieta

By Cardan:

$$3ab = m = -3 \Rightarrow ab = -1 \Rightarrow b = -\frac{1}{a}$$

$$a^3 - b^3 = n = 1 \Rightarrow a^3 + \frac{1}{a^3} = 1 \Rightarrow \left(a^3\right)^2 - a^3 + 1 = 0$$

This has no roots and so the equation can be solved by Vieta but not Cardan.

Consider the equation $x^3 - mx = n$

By Vieta the condition for a solution to be found is $27n^2 < 4m^3$

By Cardan

$$3ab = -m \Rightarrow b = -\frac{m}{3a}$$

$$a^3 - b^3 = n \Rightarrow a^3 + \frac{m^3}{27a^3} = n \Rightarrow \left(a^3\right)^2 - na^3 + \frac{m^3}{27} = 0$$

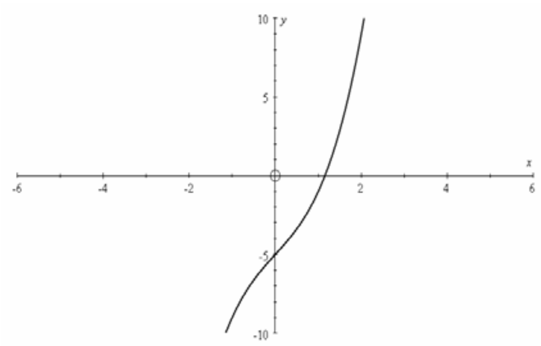
This only has roots if $n^2 > 4\frac{m^3}{27} \Rightarrow 27n^2 > 4m^3$

So all equations of this type can be solved by one method or the other.

E.g. $x^3 + 3x = 5$

For this equation $m = 3, n = 5; 27n^2 = 675, 4m^3 = 108$

$27n^2 > 4m^3$ so Cardan can be used but not Vieta.



$$3ab = 3 \Rightarrow b = \frac{1}{a}$$

$$a^3 - b^3 = 5 \Rightarrow a^3 + \frac{1}{a^3} = 5 \Rightarrow \left(a^3\right)^2 - 5a^3 - 1 = 0$$

$$\Rightarrow a^3 = \frac{5 \pm \sqrt{29}}{2} \Rightarrow a = \sqrt[3]{\frac{5+\sqrt{29}}{2}}, b = \sqrt[3]{\frac{2}{5+\sqrt{29}}} = \sqrt[3]{\frac{\sqrt{29}-5}{2}}$$

$$\Rightarrow x = \sqrt[3]{\frac{5+\sqrt{29}}{2}} - \sqrt[3]{\frac{\sqrt{29}-5}{2}} \approx 1.154171495$$

E.g. $x^3 - 3x = 5$

By Cardan

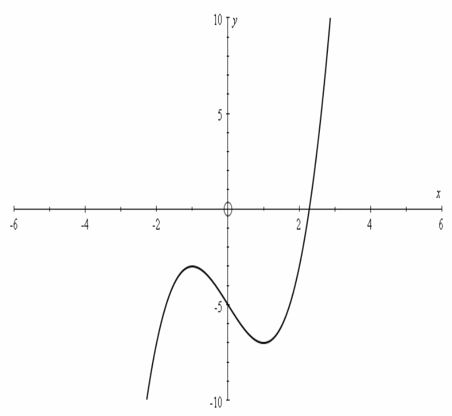
$3ab = -3 \Rightarrow b = -\frac{1}{a}$

$a^3 - b^3 = 5 \Rightarrow a^3 + \frac{1}{a^3} = 5 \Rightarrow \left(a^3\right)^2 - 5a^3 + 1 = 0$

$\Rightarrow a^3 = \frac{5 \pm \sqrt{21}}{2} \Rightarrow a = \sqrt[3]{\frac{5+\sqrt{21}}{2}}, b = -\sqrt[3]{\frac{2}{5+\sqrt{21}}} = -\sqrt[3]{\frac{5-\sqrt{21}}{2}}$

$\Rightarrow x = \sqrt[3]{\frac{5+\sqrt{21}}{2}} + \sqrt[3]{\frac{5-\sqrt{21}}{2}} \approx 2.279018786$

Note that $27n^2 = 675$ and $4m^3 = 243$ so Vieta will not work.



Example. Solve the equation. $x^3 - 4x = 2$

4. The general cubic

Consider the equation $x^3 + ax^2 + bx + c = 0$

The transformation $x = z + k$ will result in a cubic with no quadratic term for a given value of k.

$x = z + k \;\Rightarrow (z+k)^3 + a(z+k)^2 + ... = 0$

$\Rightarrow z^3 + 3kz^2 + 3k^2z + k^3 + az^2 + ... = 0$

$\Rightarrow z^3 + z^2(3k + a) + ... = 0$

So no z^2 term if $3k + a = 0 \Rightarrow k = -\dfrac{a}{3}$

E.g. $x^3 + 3x^2 + 4x - 7 = 0$

Note first that f(0) = -7, f(1) = 1 giving a first approximation of $x = 0.875$

$x = z - 1 \;\Rightarrow (z-1)^3 + 3(z-1)^2 + 4(z-1) - 7 = 0$

$\Rightarrow z^3 - 3z^2 + 3z - 1 + 3z^2 - 6z + 3 + 4z - 4 - 7 = 0$

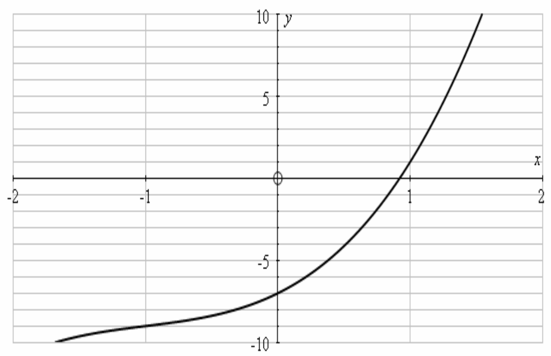
$\Rightarrow z^3 + z = 9$

$m = -1, n = 9, 27n^2 > 4m^3$ so Vieta will not work.

$3ab = 1, a^3 - b^3 = 9 \Rightarrow a^3 - \dfrac{1}{27a^3} = 9 \Rightarrow (a^3)^2 - 9a^3 - \dfrac{1}{27} = 0$

$\Rightarrow a^3 = \dfrac{9 + \sqrt{9^2 + \dfrac{4}{27}}}{2} \Rightarrow a \approx 2.0804,\; b = 0.1455$

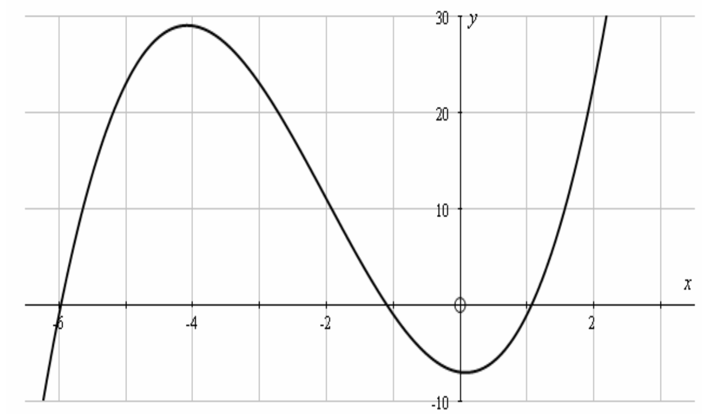
$\Rightarrow z = 1.920175121,\; x = 0.920175121$



E.g. Solve the equation $x^3 + 6x^2 - x - 7 = 0$

Newton Raphson gives the three roots to be

$x = -5.97143,\; x = -1.097255,\; x = 1.068398$



E.g. Solve the equation $x^3+6x^2-x-7=0$

Newton Raphson gives the three roots to be

$x=-5.97143,\ x=-1.097255,\ x=1.068398$

$x=z-2\Rightarrow z^3-13z=-11$

$27n^2<4m^3$ so Vieta works

$$z=ky\Rightarrow k^3y^3-13ky=11\Rightarrow 4y^3-\frac{13\times 4}{k^2}y=-\frac{44}{k^3}$$

$$k=\sqrt{\frac{52}{3}}\Rightarrow 4y^3-3y=-\frac{44}{k^3}$$

$$y=\cos\theta\Rightarrow 4\cos^3\theta-3\cos\theta=-\frac{44}{k^3}$$

$$\Rightarrow \cos 3\theta=-44\left(\frac{3}{52}\right)^{3/2}$$

$$\Rightarrow 3\theta=\arccos\left(-44\left(\frac{3}{52}\right)^{3/2}\right)\Rightarrow \theta=\frac{1}{3}\arccos\left(-44\left(\frac{3}{52}\right)^{3/2}\right)$$

$$\Rightarrow y=\cos\left(\frac{1}{3}\arccos\left(-44\left(\frac{3}{52}\right)^{3/2}\right)\right)\Rightarrow z=\sqrt{\frac{52}{3}}\cos\left(\frac{1}{3}\arccos\left(-44\left(\frac{3}{52}\right)^{3/2}\right)\right)$$

$$\Rightarrow x=\sqrt{\frac{52}{3}}\cos\left(\frac{1}{3}\arccos\left(-44\left(\frac{3}{52}\right)^{3/2}\right)\right)-2$$

and $x=\sqrt{\frac{52}{3}}\cos\left(\frac{1}{3}\left(2\pi+\arccos\left(-44\left(\frac{3}{52}\right)^{3/2}\right)\right)\right)-2$

and $x=\sqrt{\frac{52}{3}}\cos\left(\frac{1}{3}\left(2\pi-\arccos\left(-44\left(\frac{3}{52}\right)^{3/2}\right)\right)\right)-2$

5. Historical notes

Scipione dal Ferro (1465-1526) held the Chair of Arithmetic and Geometry at the University of Bologna and certainly must have met Pacioli who lectured at Bologna in 1501-2. dal Ferro is credited with solving cubic equations algebraically but the picture is somewhat more complicated. The problem was to find the roots by adding, subtracting, multiplying, dividing and taking roots of expressions in the coefficients. We believe that dal Ferro could only solve cubic equation of the form $x^3 + mx = n$. In fact this is all that is required.

However, without the Hindu's knowledge of negative numbers, dal Ferro would not have been able to use his solution of the one case to solve all cubic equations. Remarkably, dal Ferro solved this cubic equation around 1515 but kept his work a complete secret until just before his death, in 1526, when he revealed his method to his student Antonio Fior.

Fior was a mediocre mathematician and far less good at keeping secrets than dal Ferro. Soon rumours started to circulate in Bologna that the cubic equation had been solved. Nicolo of Brescia, known as Tartaglia meaning 'the stammerer', prompted by the rumours managed to solve equations of the form $x^3 + mx^2 = n$ and made no secret of his discovery.

Fior challenged Tartaglia to a public contest: the rules being that each gave the other 30 problems with 40 or 50 days in which to solve them, the winner being the one to solve most but a small prize was also offered for each problem. Tartaglia solved all Fior's problems in the space of 2 hours, for all the problems Fior had set were of the form $x^3 + mx = n$ as he believed Tartaglia would be unable to solve this type. However only 8 days before the problems were to be collected, Tartaglia had found the general method for all types of cubics.

News of Tartaglia's victory reached Girolamo Cardan in Milan where he was preparing to publish *Practica Arithmeticae* (1539). Cardan invited Tartaglia to visit him and, after much persuasion, made him divulge the secret of his solution of the cubic equation. This Tartaglia did, having made Cardan promise to keep it secret until Tartaglia had published it himself. Cardan did not keep his promise. In 1545 he published *Ars Magna* the first Latin treatise on algebra.

The method is therefore known as Cardan's method but there is little doubt that Tartaglia was the originator of the method.

See:
http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Tartaglia.html
http://mathworld.wolfram.com/CubicEquation.html
http://scienceworld.wolfram.com/biography/Tartaglia.html
http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Cardan.html
http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Viete.html
http://mathematica.ludibunda.ch/mathematicians11.html
http://mathworld.wolfram.com/VietasFormulas.html
http://mathworld.wolfram.com/CubicFormula.html